RUHR-UNIVERSITÄT BOCHUM

HOMOGENEOUS PROJECTIVE VARIETIES AND INVARIANT THEORY

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1 Introduction

The general theme of this work is the interplay between the geometry of homogeneous complex projective varieties, the structure and representation theory of their symmetry groups, and more specifically, invariant theory related to subgroups of the symmetry groups. The framework is based on the Borel-Weil theorem and the Geometric Invariant Theory, GIT, of Hilbert-Mumford. The setting is classical and the basic objects of interest are widely studied. A source of unanswered questions lies in the notorious nonconstructiveness of Hilbert's theorem asserting the existence of a finite generating set for the ring of invariants in the homogeneous coordinate ring of a complex projective variety endowed with a reductive group action. The goal of this work is to contribute to the effort for further development of the structure theory of reductive Lie groups, aiming to bound and explore the variation of certain parameters related to generating sets of invariants. Particular attention is given to the geometry of unstable loci - the zero-loci of the invariants of positive degree. Most of the results presented here are contained in the three articles [PT15], [ST18], [T19].

The present work splits into two specific topics, which have significant common ground as they both belong to the invariant theory of complex reductive groups. The first is concerned with the degrees of generating invariants on a linear representation and related geometric properties of orbits in the projectivised representation space. The second is concerned with embeddings of reductive groups and the study of invariants of the subgroup in irreducible representation of the ambient group, the resulting notion of the eigencone (generalized Littlewood-Richardson cone) and its structure; the approach is based on Geometric Invariant Theory applied to the action of the subgroup on the flag variety of the ambient group.

The first milestone on the path of both topics is the Borel-Weil theorem, providing models for the irreducible finite dimensional representations of a given connected and simply connected complex reductive group G as spaces of sections of line bundles on its flag variety $X = G/B$, where B is a Borel subgroup. Let $T \subset B \subset G$ be a fixed pair of Cartan and Borel subgroups. Let Λ denote the character lattice of T and Λ^+ denote the dominant Weyl chamber with respect to B. The irreducible finite dimensional representations of G are classified by the elements of Λ^+ as highest weights. We denote by $V(\lambda)$ the irreducible representation with highest weight $\lambda \in \Lambda^+$. On the other hand, the Picard group of X is identified with Λ, by associating the line bundle $\mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda}$ to a character $\lambda \in \Lambda$. The Borel-Weyl theorem states that

$$H^0(X, \mathcal{L}_\lambda) = V(\lambda)^* \quad \text{for} \quad \lambda \in \Lambda^+$$

and the space of sections is 0 for non-dominant λ. Thus the Weyl chamber Λ^+ is identified with the set of effective line bundles and the set of strictly dominant weights, denoted Λ^{++} - with the ample line bundles. Furthermore, the ring of sections of $\mathcal{L}_\lambda$ for $\lambda \in \Lambda^+$ is given by

$$\mathcal{R}_\lambda = \bigoplus_{j=0}^{\infty} H^0(X, \mathcal{L}_\lambda^j) = \bigoplus_{j=0}^{\infty} V(j\lambda)^* \ .$$

The Cox ring, or total coordinate ring, of X - the sum of the spaces of global section of all line bundles - is then isomorphic to the sum of all irreducible G-modules:

$$\mathrm{Cox}(X) = \bigoplus_{\mathcal{L}\in \mathrm{Pic}(X)} H^0(X, \mathcal{L}) = \bigoplus_{\lambda\in\Lambda^+} V(\lambda)\ .$$

The effective line bundles on X are in fact semiample. For a given $\lambda \in \Lambda^+$, the image $\mathbb{X}(\lambda)$ of the associated map $X \to \mathbb{P}(V(\lambda))$ is given by the G-orbit of the highest weight line $[v_\lambda] \in \mathbb{P}(V(\lambda))$, namely

$$X = G/B \to \mathbb{X}(\lambda) = G[v_\lambda] \subset \mathbb{P}(V(\lambda))\ .$$

The stabilizer of $[v_\lambda]$ is a parabolic subgroup P_λ of G containing B, and we have $\mathbb{X}(\lambda) \cong G/P_\lambda$.

Let $\Delta \subset \Lambda$ be the root system and $\Delta = \Delta^+ \sqcup \Delta^-$ be the partition into positive and negative roots corresponding to the Borel subgroup B. For any finite dimensional representation V of G, we denote by V_ν the weight space corresponding to a given $\nu \in \Lambda$ and by $\Lambda(V) \subset \Lambda$ the set of weights occurring in V. For any subset $M \subset \Lambda(V)$, we denote $V_M = \bigoplus\limits_{\nu\in M} V_\nu$. Let $W = N_G(T)/T$ denote the Weyl group. Given $\lambda \in \Lambda^+$, the weights $w\lambda$ for $w \in W$ are called the extreme weights of the irreducible G-module $V(\lambda)$. The corresponding projective points are exactly the T-fixed points in $\mathbb{X}(\lambda)$:

$$\mathbb{X}(\lambda)^T = \{x_w = [v_{w\lambda}] : w \in W\}.$$

This set is in a one-to-one correspondence with the coset space W/W_λ.

After this preparation, in the next two subsections, we present the two themes of this work and sketch the main results.

1.1 Degrees of invariant polynomials

The prototypical problem of invariant theory is the problem of describing the ring of invariant polynomials $\mathbb{C}[V]^G$, where G is a group represented linearly on a vector space V. We focus here on a classical case, where G is a connected complex semisimple algebraic group and V is finite dimensional. By Hilbert's theorem the ring of invariant polynomials $\mathbb{C}[V]^G$ is finitely generated. A significant amount of attention in the literature has been dedicated to the so-called Noether number, $\mathrm{No}(G, V)$, defined as the minimal degree d for which $\mathbb{C}[V]^G_{\leq d}$ generates the whole invariant ring. Indeed the knowledge of the Noether number reduces the determination of a generating set to a finite calculation. Note that, although the generators are not uniquely determined, the degrees of a minimal set of generators are, if one convenes to an increasing order and takes multiplicity into account. The first upper bound for the Noether number was obtained by Popov, [P81]; it depends exponentially on the input data. An upper bound depending polynomially on the input data was obtained by Derksen, [Der01]. The present work is concerned with a different problem - the invariants of low degree and, in particular, the minimal positive degree whenever it exists. We provide lower bounds for the minimal positive degree of an invariant on an irreducible representation. We also determine some numbers dividing the degrees of generators. In particular, our results yield some lower bounds for the Noether number; these bounds are unlikely to be close to the actual Noether number in general, but turn out to be exact for some special classes of representations. The methods we employ use momentum maps, combinatorics of roots and weights, and projective geometry of the variety $\mathbb{X}(\lambda)$.

We now define some key objects and formulate the two main results on this topic. Let $\lambda \in \Lambda^+ \setminus 0$ be a nontrivial dominant weight of G. Let $V = V(\lambda)$ be the corresponding irreducible representation and $\mathbb{X} = \mathbb{X}(\lambda)$ be the projective orbit of the highest weight vector. For $r \in \mathbb{N}$, the r-th secant variety of $\mathbb{X}$ is defined as

$$\Sigma_r(\mathbb{X}) = \overline{\bigcup_{x_1,...,x_r \in \mathbb{X}} \mathbb{P}Span\{x_1, ..., x_r\}}.$$

Let $J = \bigoplus_{d \geq 1} \mathbb{C}[V]_d^G$ denote the ideal in the invariant ring vanishing at 0 and let $\mathbb{P}(V)^{us}$ denote the zero-locus of J, called the unstable locus or the nullcone.

Theorem 1.1. *Let $r_{us} = \max\{r \in \mathbb{N} : \Sigma_r(\mathbb{X}) \subset \mathbb{P}(V)^{us}\}$ and, provided the invariant ring is nontrivial, let $d_1 = \min\{d \in \mathbb{N} : \mathbb{C}[V]_d^G \neq 0\}$. Then*

$$r_{us} < d_1 \; .$$

Theorem 1.2. *Suppose that $M \subset \Lambda(V)$ is a set of weights, which is linearly dependent over $\mathbb{N}$ and minimal with this property. Put $b_M = \sum_{\nu \in M} b_\nu$, where $b_\nu \in \mathbb{N}$ for $\nu \in M$ are the unique coefficients such that $\sum_{\nu \in M} b_\nu \nu = 0$ and $gcd\{b_\nu : \nu \in M\} = 1$. Then the following hold:*

(i) If $M \cap (M + \Delta) = \emptyset$, then there exists $k \in \mathbb{N}$ such that any minimal set of generators of $\mathbb{C}[V]^G$ has an element of degree kb_M.

(ii) If M consists of extreme weights, i.e. $M \subset W\lambda$, then there exists $k \in \mathbb{N}$ such that any minimal set of generators of $\mathbb{C}[V(k\lambda)]^G$ has an element of degree b_M.

The proofs and more details on this topic are given in section 3, based on the articles [PT15] and [T19], with the exception of part (ii) of the above theorem, which is proven here. We also present a variety of examples and some classes of representations for which the above procedure yields all degrees of generating invariants.

1.2 GIT for subgroup actions on flag varieties

The second theme can be viewed as a generalization of the first. It is concerned with embeddings of groups, and the study of invariants of the subgroup in irreducible representations of the ambient group. It is based on my joint work with H. Seppänen, [ST18], and presented in section 4. This introductory section contains a general discussion, motivation, and informal indications of results. Precise formulations are given in the first subsection of section 4.

We focus on connected complex semisimple algebraic groups, although some of the results are valid for reductive groups as well. A natural problem associated to a given embedding, say $\iota : \hat{G} \subset G$, is the so-called branching problem - the study of the decompositions of irreducible representations of G into irreducible $\hat{G}$-subrepresentations. This leads to the definition of the (generalized) Littlewood-Richardson monoid and cone, and their sections corresponding to invariants:

$$\begin{aligned}
&LR(\hat{G} \subset G) = \{(\hat{\lambda}, \lambda) \in \hat{\Lambda}^+ \times \Lambda^+ : (\hat{V}(\hat{\lambda})^* \otimes V(\lambda))^{\hat{G}} \neq 0\} \ , \\
&\mathcal{LR}(\hat{G} \subset G) = \mathrm{Span}_{\mathbb{R}_+} LR \subset \hat{\Lambda}_{\mathbb{R}} \times \Lambda_{\mathbb{R}} \ , \\
&LR_0(\hat{G} \subset G) = \{\lambda \in \Lambda^+ : V(\lambda)^{\hat{G}} \neq 0\} \ , \\
&\mathcal{LR}_0(\hat{G} \subset G) = \mathrm{Span}_{\mathbb{R}_+} LR_0 \subset \Lambda_{\mathbb{R}} \ .
\end{aligned}$$

The branching problem for an embedding ι can be formulated as a problem of invariants for the diagonal embedding $\hat{G} \subset \hat{G} \times G$, via the isomorphism $\mathrm{Hom}_{\hat{G}}(\hat{V}, V) \cong (\hat{V}^* \otimes V)^{\hat{G}}$. One has $LR(\hat{G} \subset G) = LR_0(\hat{G} \subset \hat{G} \times G)$. It has been shown, by Brion and Knop, that LR_0 is a finitely generated moniod, and $\mathcal{LR}_0$ is a rational polyhedral cone (cf. [E92]).

The prototypical branching problem is given by the tensor product decomposition: how does the tensor product of several, say k, irreducible representations of a given group $\hat{G}$ decompose into a sum of irreducibles? The associated embedding of groups is the diagonal embedding of $\hat{G}$ into a k-fold Cartesian product $\hat{G}^{\times k} = G$. The branching problem for a k-fold product is equivalent to the problem of invariants in a $k+1$-fold product.

Many recent advances in the study of the branching problem had been achieved by studying the problem of invariants. This is also the point of view taken here. Thus we focus on $\mathcal{LR}_0$.

The cone $\mathcal{LR}_0$ has been described using a GIT approach in a series of works, notably by Heckman, Berenstein, Sjamaar, Belkale, Kumar, Ressayre, Richmond and others (see [H82], [BS00], [BK06], [R10], [RR11] and the references therein), culminating in a description of the cone by a minimal set of inequalities, for a general embedding of reductive groups, due to Ressayre. The GIT approach is based on the identification of $\mathcal{LR}_0$ with the $\hat{G}$-ample cone on the flag variety, $C^{\hat{G}}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}}$, the closed cone generated by the ample line bundles whose section rings admit nonconstant $\hat{G}$-invariant elements. The identification

$$\mathcal{LR}_0 \cong C^{\hat{G}}(X)$$

depends on the hypothesis for $\mathcal{LR}_0$ to contain regular elements, or equivalently, for $C^{\hat{G}}(X)$ to be nonempty. This construction fits into the framework of GIT, [FMK94], thus providing tools, in particular the Hilbert-Mumford criterion, for the study of the $\hat{G}$-action and invariants. The inequalities defining $C^{\hat{G}}(X)$ in $\Lambda_{\mathbb{R}}$, given first in [BS00] and optimized in later works, are derived from the Hilbert-Mumford criterion, and have the form

$$\lambda(w\xi) \leq 0 \ , \tag{1}$$

where λ is the dominant weight representing the line bundle, w is an element of the Weyl group W of G, and ξ is an element in the lattice of one-parameter subgroups in a Cartan subgroup of $\hat{G}$ of the form $\hat{T} = \hat{G} \cap T$, identified with the integral coweight lattice $\hat{\Gamma} \subset \Gamma = \Lambda^\vee \subset \mathfrak{t}$. The relevant pairs $(\xi, w) \in \hat{\Gamma} \times W$ appearing in the inequalities are subject to certain conditions, and the description of these conditions presents the main technical issue. A finite but redundant list of pairs is given by Berenstein and Sjamaar, minimized by Belkale and Kumar in the diagonal case, and by Ressayre for arbitrary embeddings of reductive groups. The list of relevant elements ξ is relatively easy to obtain, they are determined by the weights of the $\hat{G}$-action on the quotient of Lie algebras $\mathfrak{g}/\hat{\mathfrak{g}}$. In the diagonal case, one has simply the fundamental coweights of $\hat{G}$. The relevant Weyl group elements present a more delicate problem. The conditions on w, given in the aforementioned series of works starting with [BS00], are cohomological, stated in terms of pullbacks of Schubert classes from flag varieties of G to closed $\hat{G}$-orbits in them. There is an interest in a cohomology-free description of the $\hat{G}$-ample cone, and this has been achieved for diagonal embeddings in [BK06] with a non optimal list, optimized for some classical groups in terms of quiver representations by Derksen-Weyman [DW11] for groups of type A, and by Ressayre [R12] in types A,B,C.

One of our results is a cohomology free description of the cone $C^{\hat{G}}(X)$ by a finite list of inequalities. The list is redundant in general, and in many cases difficult to compute. However, it has a qualitative flavour, which allows for some interesting and new structural properties of the eigencone to be drawn. A formulation is given in Theorem 4.13. Below we give a formula for the $\hat{G}$-ample cone, which is relatively simple to express, but yields an infinitude of inequalities. The cohomological condition is replaced by a dimension condition for a subvariety of X, the saturation of a Schubert variety, associated to a pair (ξ, w). We derive the following description of the $\hat{G}$-ample cone, in case it is nonempty:

$$\begin{aligned} C^{\hat{G}}(X) = \{ \lambda \in \Lambda^+_{\mathbb{R}} :\ & \lambda(w^{-1}\xi) \leq 0 \\ & \text{for all pairs } (\xi, w) \in \hat{\Gamma}^+ \times W \text{ such that} \\ & \dim \hat{G} P_\xi x_w = \dim \hat{G}/\hat{P}_\xi + \dim P_\xi x_w = \dim G/B \ \} \end{aligned}$$

where $\hat{P}_\xi \subset \hat{G}$ and $P_\xi \subset G$ are the parabolic subgroups defined by ξ, and $x_w = wB \in X$. Concering computability, the dimensions of $\hat{G}/\hat{P}_\xi$ and $P_\xi x_w$ can be computed combinatorially, in terms of root systems and Weyl groups. The inequality $\dim \hat{G} P_\xi x_w \leq \dim \hat{G}/\hat{P}_\xi + \dim P_\xi x_w$ follows directly from the definitions, but the condition for equality is subtle, and presents one of the main technical issues in this line of work. Our proof of the above description of $C^{\hat{G}}(X)$ is in fact parallel to that of [BS00], the difference being rather formal than essential, but we present a full argument based directly on the Hilbert-Mumford criterion, because our key step - the closed formula for the unstable locus - is not to be found elsewhere in an explicit form to the best of our knowledge, and is needed for the rest of our study. Our goal is in fact the structure behind the boundary of $C^{\hat{G}}(X)$.

For a view on the global behaviour of invariants, it is convenient to consider the Cox ring of X. The $\hat{G}$-invariants we are after are all assembled in the invariant ring $\mathrm{Cox}(X)^{\hat{G}}$, which is also finitely generated. Cox rings are an important ingredient in the theory of Mori dream spaces, the latter having finitely generated Cox rings as one of their essential defining properties (cf. [HK00] for the full definition). The flag varieties form indeed a class of known examples. It is natural to ask about a variety, a quotient Y, with $\mathrm{Cox}(X)^{\hat{G}}$ as a Cox ring, having the classical result for individual line bundles in mind. Such a variety would be a geometric incarnation of the complete information on invariants for the given pair $\hat{G} \subset G$. This topic is addressed in [S14], where such quotients are constructed and shown to be Mori dream spaces. The construction rests, however, on a nontrivial assumption for existence of $\hat{G}$-movable chambers in $C^{\hat{G}}(X)$. The latter consists of line bundles whose rings of nontrivial invariants have vanishing locus, the unstable locus $X^{us}(\lambda)$, of codimension at least 2 in X, containing all points with positive dimensional stabilizers. This motivates the study of the GIT-classes of line bundles - the equivalence classes defined by equality of the unstable locus.

We study the GIT-classes for the $\hat{G}$-action on X, and we address specifically the question of existence of $\hat{G}$-movable chambers. We devise a general method built on our closed formula for the unstable locus. This formula allows us to study GIT-classes of line bundles and their variations. We give a description of the GIT-classes, showing that all inequalities defining chambers in $C^{\hat{G}}(X)$ are of same type as (1), and we provide a procedure arriving at the relevant ξ, w, formulated in Theorem II and Theorem 4.10 in section 4. We show that the codimension of the unstable locus is equal to 1 at the regular boundary of the $\hat{G}$-ample cone grows in steps of 1 towards the interior, so that there is a sequence of convex polyhedral cones $C^{\hat{G}}(X) = C_1 \supset C_2 \supset C_3 \ldots$, where C_j is spanned by line bundles with codimension of the unstable locus at least j. We derive a criterion for existence of $\hat{G}$-movable chambers in terms of the structure of the embedding $\hat{G} \subset G$. For diagonal embeddings $\hat{G} \subset \hat{G}^{\times k} = G$, we establish the existence of $\hat{G}$-movable chambers for sufficiently large k.

Another one of the main results, formulated as Theorem IV in section 4, concerns the birational geometry of a GIT-quotient Y of X, by a $\hat{G}$-movable chamber. We establish a canonical identification of the GIT-chambers in $C^{\hat{G}}(X)$ with the Mori chambers in the pseudoeffective cone in $\mathrm{Pic}(Y)$. In particular, we show that

$$\mathrm{Cox}(X)^{\hat{G}} \cong \bigoplus_{\lambda \in \Lambda^+} V(\lambda)^{\hat{G}} \cong \text{Finite extension of } \mathrm{Cox}(Y) \; .$$

Precise statements are given in section 4.

2 Preliminaries: the Hilbert-Mumford criterion and the Kirwan-Ness stratification

The following result of Hilbert is fundamental in invariant theory and a basic ingredient for the present work. It reduces the instability for linear actions of reductive groups to instability for their one-parameter subgroups. It has been developed further by Mumford. We refer to [FMK94], for the general theory, and [N84] for a shorter presentation suitable for our purposes. In this section we recall the basic results we need.

Hilbert's theorem: *Let $G \to GL(V)$ be a representation of a reductive complex algebraic group G. Then the ring of invariants $\mathbb{C}[V]^G$ is generated by a finite number of homogeneous elements. Let $J \subset \mathbb{C}[V]^G$ be the ideal vanishing at 0 and let $\mathbb{P}(V)^{us}_G \subset \mathbb{P}(V)$ denote its zero locus, called the G-unstable locus. Then*

$$\begin{aligned}\mathbb{P}(V)^{us}_G &= \{[v] \in \mathbb{P}(V) : \overline{Gv} \ni 0\} \\ &= \{[v] \in \mathbb{P}(V) : \exists \gamma \in \mathrm{Hom}(\mathbb{C}^*, G) : \lim_{t\to 0} \gamma(t)v = 0\} .\end{aligned}$$

More generally, let Z be a smooth G-variety with a G-equivariant ample line bundle $\mathcal{L}$ defining an embedding $Z \subset \mathbb{P}(V)$. Let

$$J_{\mathcal{L}} = \bigoplus_{j\geq 1}^{\infty} H^0(Z, \mathcal{L}^j)^G$$

The unstable locus in Z with respect $\mathcal{L}$ is defined as

$$Z^{us}(\mathcal{L}) = Z^{us}_G(\mathcal{L}) = Z(J_{\mathcal{L}}) = Z \cap \mathbb{P}(V)^{us}_G .$$

The semistable locus is the complement of the unstable: $Z^{ss}(\mathcal{L}) = Z \setminus Z^{us}(\mathcal{L})$. The GIT-quotient of Z with respect to the G-action on $\mathcal{L}$ is defined by Hilbert's equivalence relation as

$$Y = Z^{ss}(\mathcal{L})//G \quad , \quad \text{where} \quad x \sim y \Longleftrightarrow \overline{Gx} \cap \overline{Gy} \cap Z^{ss}(\mathcal{L}) \neq \emptyset .$$

Theorem 2.1. *(cf. [FMK94]) There exist a line bundle $\underline{\mathcal{L}}$ on Y and a number $q \in \mathbb{N}$, such that for every $j \in \mathbb{N}$ there is an isomorphism*

$$H^0(Y, \underline{\mathcal{L}}^j) \cong H^0(Z, \mathcal{L}^{qj})^G .$$

Mumford has devised a numerical criterion for instability for equivariant ample line bundles on projective varieties. We identify the elements $\gamma \in \mathrm{Hom}(\mathbb{C}^*, G)$ with their infinitesimal generators in the Lie algebra $\xi = \dot{\gamma}(1) \in \mathfrak{g}$ and call them one-parameter subgroups (OPS). The OPS of the Cartan subgroup $T \subset G$ form a lattice naturally identified with the dual to the weight lattice $\Gamma = \Lambda^\vee \subset \mathfrak{t}$. Recall that every OPS of G is conjugate to a unique element of Γ^+, the set of dominant elements with respect to the Borel subgroup B. We consider the ξ-unstable locus, taking the orientation into account:

$$Z^{us}_\xi(\mathcal{L}) = \{[v] \in Z : \lim_{t\to -\infty} \exp(t\xi)v = 0\}.$$

Mumford's numerical function for $\xi \in \Gamma$,

$$M^\xi : Z \to \mathbb{Z} , \tag{2}$$

is defined as follows. For $x \in Z$ let $x_0 = \lim_{t\to -\infty} \exp(t\xi)x \in Z$. The limit point belongs to the fixed set of the OPS, $x_0 \in Z^\xi$. The connected components of

Z^ξ are contained in the projectivizations of the eigenspaces of ξ. Define $M^\xi(x)$ to be the eigenvalue of ξ at x_0. The point is ξ-unstable if the eigenvalue is positive.

Hilbert-Mumford criterion: *Let $\mathcal{L}$ be a G-equivariant ample line bundle on Z. A point $x \in Z$ is G-unstable if and only if it is unstable for some one-parameter subgroup of G. We have*

$$Z^{us}_G(\mathcal{L}) = GZ^{us}_T(\mathcal{L}) = G\left(\bigcup_{\xi\in\Gamma^+} Z^{us}_\xi(\mathcal{L})\right) \quad , \quad Z^{us}_\xi(\mathcal{L}) = \{x \in Z : M^\xi(x) > 0\}.$$

For any $x \in Z$, let $\tilde{x} \in V$ be a vector with $[\tilde{x}] = x \in Z \subseteq \mathbb{P}(V)$. We can decompose $\tilde{x}$ as a sum of weight vectors,

$$\tilde{x} = \sum_{\nu\in\Lambda(V)} v_\nu. \tag{3}$$

For $x \in Z$, let $St(x) \subseteq \Lambda(V)$ denote the set of weights ν for which $v_\nu \neq 0$ in the decomposition (3). Then we have

$$M^\xi(x) = \min\{\nu(\xi) : \nu \in St(x)\}\ . \tag{4}$$

To compute the dimensions of the unstable loci, we shall need some general results from geometric invariant theory concerning stratifications of unstable loci. Specifically, the stratification theorems due to Kirwan in the symplectic setting and Ness for projective varieties, see [Kir84], sections 12 and 13, and [N84], Theorem 9.5. More recently, Popov, [P03], has refined the stratification results for (projective) representation spaces. It turns out that Popov's constructions can be applied successfully for complete flag varieties as well, as we show in Section 4.5.

The so-called Hesselink strata of Z^{us}, as they are defined in [N84] for any equivariantly embedded smooth G-variety $Z \subset \mathbb{P}(V)$ with a G-linearized $\mathcal{O}(1)$, have the form $GZ_{\xi,m}$, where $Z_{\xi,m}$ is the so-called blade, determined by a one-parameter subgroup $\xi \in \Gamma$ and a positive integer m obtained as the value of a weight of a ξ-fixed point on Z. Formally, any $\xi \in \Gamma$ defines a eigenspace decomposition

$$V = \bigoplus_{m\in\mathbb{Z}} V^{\xi,m}\ , \quad \text{where} \quad V^{\xi,m} = \{v \in V : \xi v = mv\}\ .$$

The fixed point set in Z is then partitioned as

$$Z^\xi = \bigsqcup_{m\in\mathbb{Z}} Z^{\xi,m}\ , \quad \text{where} \qquad Z^{\xi,m} = Z \cap \mathbb{P}(V^{\xi,m})\ .$$

The blade $Z_{\xi,m}$ is defined as set set of points flowing into $Z^{\xi,m}$ under $\xi_t = \exp(t\xi)$ as $t \to -\infty$. In the projective situation the blades are obtained by intersection with the blades of the ambient projective space and are given by (cf. [P03])

$$Z_{\xi,m} = Z \cap \mathbb{P}(V)_{\xi,m}\ , \quad \text{with} \quad \mathbb{P}(V)_{\xi,m} = \mathbb{P}(V^{\xi\geq m}) \setminus \mathbb{P}(V^{\xi>m})\ ,$$

where $V^{\xi \geq m}$ denotes the sum of the eigenspaces with eigenvalue greater or equal to m and similarly for $V^{\xi > m}$. Note that $\mathbb{P}(V)_{\xi,m}$ is an orbit of the parabolic subgroup of $SL(V)$ defined by ξ and the limit set $\mathbb{P}(V^{\xi,m})$ is an orbit of its Levi subgroup $SL(V)_\xi$, the centralizer of ξ. The blade $Z_{\xi,m}$ is preserved by the parabolic subgroup P_ξ and the limit set $Z^{\xi,m}$ is preserved by the centralizer of ξ, which is a Levi subgroup denoted by L_ξ. There is a natural surjective map

$$G \times_{P_\xi} Z_{\xi,m} \to GZ_{\xi,m} \ .$$

By the Hilbert-Mumford criterion, Z^{us}_G can be written as the union of $GZ_{\xi,m}$ over $\xi \in \Gamma^+$ and $m > 0$. The stratification theorems concern the existence of a finite number of blades, whose G-saturation gives the entire unstable locus. Kirwan gives a characterization of these stratifying blades, referring to the connected components of $Z^{\xi,m}$ and suitable sets of semistable points in them.

Definition 2.1. *A dominant one-parameter subgroup $\xi \in \Gamma^+$ is called a stratifying element for Z^{us}_G, if it satisfies the following conditions:*

(i) ξ is indivisible, i.e., $\frac{1}{k}\xi \notin \Gamma$ for all $k > 1$;

(ii) there exists $m > 0$ such that $(Z^{\xi,m})^{ss}_{L_\xi/\xi}(\mathcal{O}(1)) \neq \emptyset$,

The pairs (ξ, m) for which (ii) holds are called stratifying pairs, and the corresponding blades $Z_{\xi,m}$ - stratifying blades. We denote by $\mathfrak{S} \subset \Gamma^+$ the set of the stratifying elements.

Note that the first condition is necessary just to remove the obvious redundancy arising from $Z^{\xi,m} = Z^{k\xi,km}$, while the second condition contains the essence of the notion. The next theorem states that the stratifying OPS are obtained as follows. For any set of weights $S \in \Lambda$, consider the closest to zero point $\nu_S \in \mathrm{Conv}(S) \subset \Lambda_\mathbb{R}$ and let $\xi_S \in \Gamma$ denote the indivisible OPS generating the ray in $\mathfrak{h}$ corresponding, under the Killing from, to the ray of ν_S. We set $\xi_S = 0$ if $\nu_S = 0$. Similarly, if $Z_1 \subset Z \subset \mathbb{P}(V)$ is a subvariety preserved by T, we denote by $St(Z_1^T) \subset \Lambda$ the set of weights of the T-fixed set, and by $\xi_{Z_1} = \xi_{St(Z_1^T)} \in \Gamma$ the OPS resulting from this set of weights.

Theorem 2.2. (Kirwan, [Kir84], Ness, [N84])

Let $Z \subset \mathbb{P}(V)$ be a smooth projective variety preserved by a reductive group G linearly represented on V. Let $St(Z^T)$ be the set of weights of the fixed point set of a Cartan subgroup $T \subset G$. Let $\Xi = \{\xi_S : S \subset St(Z^T)\} \cap (\Gamma^+ \setminus \{0\})$. Then

$$Z^{us}_G = \bigsqcup_{\xi \in \Xi, m \in \mathbb{N}} G(Z_{\xi,m})^{ss}_{L_\xi/\xi} \ .$$

For a nonempty stratum, the natural map $G \times_{P_\xi} (Z_{\xi,m})^{ss}_{L_\xi/\xi} \to G(Z_{\xi,m})^{ss}_{L_\xi/\xi}$ is finite, and the dimension of the stratum is

$$\dim G(Z_{\xi,m})^{ss}_{L_\xi/\xi} = \dim G/P_\xi + \dim Z_{\xi,m} \ .$$

Remark 2.1. *The union in the formula is disjoint, but the index set, as written above, gives rise to empty summands. The obvious redundancy comes form m: clearly the values of the weights of V on any fixed ξ are bounded since V is finite dimensional. A more subtle problem is presented by empty summands arising from blades with empty semistable locus, and we shall see these manifest when Z is a flag variety.*

2.1 Momentum maps

There is an alternative approach to GIT using symplectic geometry and momentum maps, yielding an identification of the GIT-quotient with a symplectic reduction space. We sketch it here for the case of the projective space $\mathbb{P} = \mathbb{P}(V)$ of a G-module V. Let $H \subset T$ be the maximal compact subgroup of the given torus $T \subset G$, and let $K \subset G$ be a maximal compact subgroup of G containing H. Let $\langle , \rangle$ be a K-invariant Hermitean form on V. The Killing form (in the semisimple case, and any invariant form in the reductive case) on $\mathfrak{k}$ defines an isomorphism between $\mathfrak{k}$ and $\mathfrak{k}^*$, and allows to embed $\mathfrak{h}^*$ as a subspace of $\mathfrak{k}^*$.

We consider the K-equivariant momentum map:

$$\mu : \mathbb{P} \longrightarrow i\mathfrak{k}^*,\ \mu[v](\xi) = \frac{\langle \xi v, v\rangle}{\langle v, v\rangle},\ [v] \in \mathbb{P}, \xi \in \mathfrak{k}.$$

We denote the fibres of the momentum by $\mathcal{M}_\xi = \mu^{-1}(\xi)$. Then $\mathcal{M}_0$, if non-empty, is preserved by K, contained in the semistable locus and, by a theorem of Kirwan, one has $\mathcal{M}_0/K = \mathbb{P}^{ss}//G$, where the latter denotes the GIT-quotient. The Kirwan-Ness stratification can be obtained via Morse-theoretic methods using the function $||\mu||^2$, cf. [Kir84].

3 Degrees of generating invariants and projective geometry

Let G be a connected complex reductive algebraic group and V be a finite dimensional G-module. By Hilbert's theorem the invariant ring $\mathbb{C}[V]^G$ is finitely generated and the generators can be chosen homogeneous. The finite set of generators is not unique, but the degrees of a minimal set of generators are uniquely determined, if one convenes to an increasing order and takes multiplicity into account. We say that $\mathbb{C}[V]^G$ admits a generator of degree $d > 0$, if the degree component $\mathbb{C}[V]^G_d$ is not contained in the subring generated by $\mathbb{C}[V]^G_{<d}$. The maximal degree of a generator is called the Noether number, $\mathrm{No}(G, V)$, this is the minimal d for which $\mathbb{C}[V]^G_{\leq d}$ generates $\mathbb{C}[V]^G$. If $\mathbb{C}[V]^G \neq \mathbb{C}$, we denote by d_1 the minimal positive degree of an invariant polynomial.

From now on, by *a generator* of $\mathbb{C}[V]^G$ we mean *an element of a minimal set of generators.*

Recall that we have a fixed Cartan subgroup $T \subset G$, and for a given set of weights $M \subset \Lambda(V)$ we denote by V_M the sum of the corresponding weight spaces. We also denote $\mathbb{P} = \mathbb{P}(V)$ and $\mathbb{P}_M = \mathbb{P}(V_M)$.

Definition 3.1. *A subset $M \subset \Lambda$ is called root-distinct if $M \cap (\Delta + M) = \emptyset$.*

Definition 3.2. *A subset $M \subset \Lambda$ is called a balanced simplex, if it is linearly dependent over $\mathbb{Z}_{>0}$ and minimal with this property.*

For any balanced simplex M, there exists unique positive integers $b_\nu, \nu \in M$ such that $\sum\limits_{\nu \in M} b_\nu \nu = 0$ and $gcd\{b_\nu : \nu \in M\} = 1$; we denote

$$b_M = \sum_{\nu \in M} b_\nu \,.$$

Theorem 3.1. *Let $G \to GL(V)$ be a finite dimensional representation. Suppose that $M \subset \Lambda(V)$ satisfies the following two properties:*

(i) *M is root-distinct;*

(ii) *M is a balanced simplex.*

Then the ring of invariants $\mathbb{C}[V]^G$ admits a generator of degree kb_M for some integer $k \geq 1$.

Proof. We shall need the following lemma concerning the momentum map $\mu : \mathbb{P} \to i\mathfrak{k}^*$ defined in Section 2.1.

Lemma 3.2. *(Wildberger) If $M \subset \Lambda(V)$ is a root-distinct set of weights, then*

$$\mu(\mathbb{P}_M) = \mathrm{Conv}(M) \subset i\mathfrak{h}^*.$$

In fact, Wildberger, [Wi92], proved a similar statement for some very special sets of weights, namely Weyl group orbits, but his method goes through in the general case, as we show below.

Proof. Let $\mathfrak{g} = \mathfrak{t} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$ be the root space decomposition with respect to T and let Δ^+ be the system of positive roots for a fixed Borel subgroup $B \subset G$ containing T. It is well known that the root vectors $e_\alpha \in \mathfrak{g}_\alpha$ can be chosen so that $\mathfrak{k}$ is spanned by $\mathfrak{h}$ and $e_\alpha - e_{-\alpha}, i(e_\alpha + e_{-\alpha})$ for $\alpha \in \Delta^+$. The defining formula for μ clearly extends to a map $\mathbb{P} \to \mathfrak{g}^*$ and we denote the coordinate functions, for $\xi \in \mathfrak{g}$, by $\mu^\xi : \mathbb{P} \to \mathbb{C}$, $\mu^\xi[v] = \langle \xi v, v \rangle / \langle v, v \rangle$. It is easy to see that the simultaneous vanishing of $\mu^{e_\alpha - e_{-\alpha}}$ and $\mu^{i(e_\alpha + e_{-\alpha})}$ is equivalent to the vanishing of μ^{e_α} and $\mu^{e_{-\alpha}}$. Hence $\mu[v] \in i\mathfrak{h}^*$ if and only if $\mu^{e_\alpha}[v] = 0$ for all α.

The root vectors send weight spaces to weight spaces: $e_\alpha(V_\nu) \subset V_{\nu+\alpha}$. For a given nonzero $v \in V$, the orthogonal projections to the weight spaces, $\mathrm{pr}_\nu : V \to V_\nu$, define a unique decomposition as a sum of weight vectors $v = \sum \mathrm{pr}_\nu(v)$; the set $\mathrm{St}(v) = \{\nu \in \Lambda(V) : \mathrm{pr}_\nu(v) \neq 0\}$ is called the support of v. Now observe that $\mathrm{St}(v)$ is a root-distinct set if and only if $\mathrm{St}(v) \cap \mathrm{St}(e_\alpha v) = \emptyset$ for all $\alpha \in \Delta$. In such a case, the orthogonality of weight spaces implies $\mu^{e_\alpha}[v] = 0$ for all α, and by the above remarks we get $\mu[v] \in i\mathfrak{h}^*$. In fact $\mu[v] = \mu_H[v]$, where μ_H denotes the momentum map for the H-action, which is equal to the composition of μ with the orthogonal projection from $i\mathfrak{k}^*$ to $i\mathfrak{h}^*$. We can conclude that, for a given root-distinct set $M \in \Lambda(V)$, we have

$$\mu(\mathbb{P}_M) = \mu_H(\mathbb{P}_M).$$

The latter is equal to $\mathrm{Conv}(M)$ by the well known theorem of Atiyah for momentum maps of tori, [A82]. However, for the case at hand, the direct calculation of $\mu_H[v]$ is also easily accessible. We may assume that $||v|| = 1$. For $\nu \in \mathrm{St}(v)$, put $a_\nu = ||\mathrm{pr}_\nu(v)||$ and $v_\nu = \frac{1}{a_\nu}\mathrm{pr}_\nu(v)$. Then

$$\mu_H[v] = \mu_H[\sum_{\nu \in \mathrm{St}(v)} a_\nu v_\nu] = \sum_{\nu \in M} |a_\nu|^2 \nu, \quad \text{with} \quad \sum_{\nu \in M} |a_\nu|^2 = 1.$$

This implies $\mu(\mathbb{P}_M) = \mathrm{Conv}(M) \subset i\mathfrak{h}^*$. □

Let us return to the proof of the theorem. By the above lemma, the hypothesis (i) implies that $\mu(\mathbb{P}_M) = \mathrm{Conv}(M)$. From hypothesis (ii) we infer that $0 \in \mathrm{Conv}(M)$, and hence $\mathcal{M}_0 \cap \mathbb{P}_M \neq \emptyset$. In particular, picking any set of weight vectors $v_\nu \in V_\nu$ of norm 1, and setting

$$v = \sum_{\nu \in M} \sqrt{b_\nu} v_\nu, \quad \text{we get} \quad \mu[v] = \frac{1}{||v||} \sum_{\nu \in M} b_\nu \nu = 0.$$

By Heckman's theorem, [H82], it follows that $[v] \notin \mathbb{P}^{ss}$ and hence there exists a nonconstant homogeneous invariant polynomial $f \in \mathbb{C}[V]^G$ with $f(v) \neq 0$. The restriction of f to the torus orbit $Tv \subset V$ is a nonzero constant. The orbit closure $\mathbb{L} = \overline{T[v]} \subset \mathbb{P}_M$ is a projective toric T-variety and, accidentally, a linear subspace of $\mathbb{P}$. Let R denote the homogeneous coordinate ring of $\mathbb{L}$ and let $\mathrm{res} : \mathbb{C}[V] \to R$ denote the quotient morphism, which is surjective and T-equivariant. We have $\mathrm{res}(\mathbb{C}[V]^G) \subset R^T$; this restricted map is not necessarily surjective, but we have $\mathrm{res}(f) \neq 0$. Since $\mathbb{L}$ is a toric variety, R^T is isomorphic to a polynomial ring on one variable. The hypothesis (ii) and specifically the supplementary assumption $\gcd\{b_\nu : \nu \in M\} = 1$ implies that R^T is generated by $\mathrm{res}(p)$, where

$$p = \prod_{\nu \in M} z_\nu^{b_\nu}, \quad \deg(p) = b_M,$$

where z_ν denotes the coordinate to v_ν. Indeed, p is T-invariant, it does not vanish on v, it is not a power of any other polynomial, and there is no T-invariant monomial of smaller degree in the variables $z_\nu, \nu \in M$.

We have $\mathrm{res}(f) \neq 0$, hence $\mathrm{res}(f) = \mathrm{res}(p)^k$ for some k, and $\deg(f) = kb_M$. □

Example 3.1. *(The adjoint representation of a simple group)*

Consider the case where G is simple of rank ℓ and $V = \mathfrak{g}$ is the adjoint representation. It is well known that $\mathbb{C}[\mathfrak{g}]^G$ is isomorphic to a polynomial ring in ℓ variables. The degrees $d_1, ..., d_\ell$ of the generators are also well known, and are related to a variety of important objects associated to G. The minimal degree is 2 and the maximal is the Coxeter number h of G.

The set of weights with respect to a Cartan subgroup T is $\Delta \cup \{0\}$. Let Π be the set of simple roots with respect to a fixed Borel subgroup B and let $\theta = \sum_{\alpha\in\Pi} m_\alpha \alpha$ be the highest root, which is also the highest weight of $\mathfrak{g}$. The Coxeter number, which we also denote by h_Π when necessary, is the given by

$$h = h_\Pi = 1 + \sum_{\alpha\in\Pi} m_\alpha.$$

Denote $\Pi^Q = \Pi \cup \{-\theta\}$. This set is a root-distinct balanced simplex with

$$b_{\Pi^Q} = 1 + \sum_{\alpha\in\Pi} m_\alpha = h_\Pi.$$

More generally, consider any subset $\tilde{\Pi} \subset \Pi$ corresponding to a simple subgroup of G, i.e., to a connected subdiagram of the Dynkin diagram. We denote $\tilde{\Pi}^Q = \tilde{\Pi} \cup \{-\tilde{\theta}\}$, where $\tilde{\theta}$ is the highest root of the sub-root-system generated by $\tilde{\Pi}$. Then $\tilde{\Pi}^Q$ is a root-distinct balanced simplex in Λ, and $b_{\tilde{\Pi}^Q} = h_{\tilde{\Pi}}$ is the Coxeter number of the corresponding simple root system. Hence there is an invariant generator of degree $\tilde{q} h_{\tilde{\Pi}}$ for some $\tilde{q} \in \mathbb{N}$. A connected Dynkin diagram with ℓ nodes admits connected subdiagrams of with $\tilde{\ell}$ nodes for any $1 \le \tilde{\ell} \le \ell$. Since $\#\tilde{\Pi}^Q = \#\tilde{\Pi} + 1$, the root system Δ admits root-distinct balanced simplices of all dimensions from 1 to ℓ. Using the known values Coxeter number and the degrees of generating invariants, one finds out that this procedure yields all generators. An a priori proof of this fact could be of interest.

3.1 Secant varieties and a lower bound on the degrees

We assume now that $V = V(\lambda)$ is an irreducible representation of a semisimple group G, with $\lambda \in \Lambda^+$ being the highest weight of V with respect to a fixed pair of Cartan and Borel subgroups $T \subset B \subset G$. We let $\mathbb{X} = G[v_\lambda] \subset \mathbb{P} = \mathbb{P}(V)$ be the orbit of the highest weight line, the unique closed projective G-orbit. We also assume that $\lambda \neq 0$, so that the representation is nontrivial.

We consider linear combinations of points from the affine cone over $\mathbb{X}$. Since the representation is irreducible the variety $\mathbb{X}$ spans the ambient space. The rank function on $\mathbb{P}$ with respect to $\mathbb{X}$ is defined as

$$\mathrm{rk}_\mathbb{X} : \mathbb{P} \longrightarrow \mathbb{N}, \quad \mathrm{rk}_\mathbb{X}[v] = \min\{r \in \mathbb{N} : v = x_1 + ... + x_r, [x_j] \in \mathbb{X}\}.$$

The rank subsets of $\mathbb{P}$ are defined as

$$\mathbb{X}_r = \{[v] \in \mathbb{P} : \mathrm{rk}_\mathbb{X}[v] = r\}.$$

The r-th secant variety of $\mathbb{X}$ is defined as the Zariski closure of the union of linear spaces spanned on r points of $\mathbb{X}$:

$$\Sigma_r = \Sigma_r(\mathbb{X}) = \overline{\bigcup_{x_1,...,x_r\in\mathbb{X}} \mathbb{P}\mathrm{Span}(x_1, ..., x_r)} = \overline{\bigcup_{s\le r} \mathbb{X}_s}.$$

The border rank on $\mathbb{P}$ with respect to $\mathbb{X}$ is defined as

$$\underline{\mathrm{rk}}_{\mathbb{X}} : \mathbb{P} \leq \mathbb{N}, \quad \underline{\mathrm{rk}}_{\mathbb{X}}[v] = \min\{r \in \mathbb{N} : [v] \in \Sigma_r\}.$$

There is a unique integer $r_g \geq 1$ for which $\mathbb{X}_{r_g}$ is open in $\mathbb{P}$ and this is the smallest r for which $\Sigma_r = \mathbb{P}$; r_g is called the generic rank. We also denote by $r_{\max}$ the maximal value of $\mathrm{rk}_{\mathbb{X}}$. The rank function is G-invariant, and consequently the sets $\mathbb{X}_r$ and Σ_r are preserved by G. We have containments of varieties $\mathbb{X} \subset \Sigma_2 \subset ... \subset \Sigma_{r_g} = \mathbb{P}$ and there is a corresponding chain of G-stable ideals $I(\mathbb{X}) \supset I(\mathbb{X}_2) \supset ... \supset 0$. The ideal of $\mathbb{X}$ is generated in degree 2, by a suitable generalization of the Plücker relations, due to Kostant, see e.g. [Lan12]. The following theorem describes the first nonzero degree component of the ideal of the r-th secant variety of a variety cut out by quadrics. We formulate it here for the case in hand. We use the identification of $\mathbb{C}[V]$ with the space of symmetric tensors SV^* inside the tensor algebra on V^*.

Theorem 3.3. (Landsberg and Manivel, [LM03])
The first nonzero homogeneous component of the ideal $I(\Sigma_r)$ is in degree $r+1$ and is given, for $r \geq 2$, by the $(r-1)$-st prolongation of the generating space $I_2(\mathbb{X})$ of the ideal of $\mathbb{X}$, i.e.

$$I_r(\Sigma_r) = 0, \quad I_{r+1}(\Sigma_r) = S^{r+1}V^* \cap (I_2(\mathbb{X}) \otimes S^{r-1}V^*).$$

Definition 3.3. *The rank of instability of the irreducible representation $V = V(\lambda)$ is defined as*

$$r_{us} = \max\{r \in \mathbb{N} : \Sigma_r \subset \mathbb{P}^{us}\}.$$

When $\mathbb{C}[V]^G \neq \mathbb{C}$, so that the semistable locus is nonempty, the rank of semistability is defined as

$$r_{ss} = \min\{r \in \mathbb{N} : \Sigma_r \subset \mathbb{P}^{ss} \neq \emptyset\} = r_{us} + 1.$$

Remark 3.1. *1) The closed G-orbit $\mathbb{X} \subset \mathbb{P}$ belongs to $\mathbb{P}^{us}$ as long as the representation is nontrivial; thus $r_{us} \geq 1$.*

2) The incidence with the nullcone for a projective variety can be tested via the momentum map with respect to a maximal compact subgroup $K \subset G$ and an invariant Hermitean form on V. We have: $\Sigma_r \subset \mathbb{P}^{us}$ if and only if $0 \notin \mu(\Sigma_r)$.

The following proposition is an interpretation of a result of Zak, [Z93], Ch. III.

Proposition 3.4. *If $V(\lambda) \ncong V(\lambda)^*$, then $\Sigma_2 \subset \mathbb{P}^{us}$. If $\mathbb{C}[V(\lambda)]^G \neq \mathbb{C}$, then $r_{ss} = 2$ if and only if $V(\lambda) \cong V(\lambda)^*$.*

The above results have the following direct consequence.

Theorem 3.5. *If a nonconstant homogeneous invariant $f \in \mathbb{C}[V(\lambda)]^G$ vanishes on Σ_r, then $deg(f) > r$.*

Suppose that $\mathbb{C}[V(\lambda)]^G \neq \mathbb{C}$ and let d_1 be the minimal positive degree of a homogeneous invariant polynomial. Then

$$r_{ss} \leq d_1.$$

In view of the above theorem, it is natural to ask: given $\Sigma_r \cap \mathbb{P}^{ss} \neq 0$, is there indeed an invariant of degree r? Such an invariant may or may not appear (see Example 3.2) but the relation between rank and degree is not accidental. It stems from the fact that certain monomials of invariant polynomials have the form $(z_1...z_r)^k$, where z_j are coordinates with respect to vectors $x_1, ..., x_r$ in the affine cone $\hat{\mathbb{X}}$, taken as a part of a basis in $V(\lambda)$. The construction is based on Theorem 3.1.

Recall that

$$\mathbb{X}^T = \{x_w = [v_{w\lambda}] : w \in W\}.$$

Corollary 3.6. *Let $\lambda \in \Lambda^+$ and $\mathbb{X} = G[v_\lambda] \subset \mathbb{P}(V(\lambda))$. If the Weyl group orbit $W\lambda$ contains a root-distinct balanced simplex with r-elements, then $\mathbb{C}[V(\lambda)]^G$ admits a generator which does not vanish on the secant variety $\sigma_r(\mathbb{X})$ and $r \geq r_{ss}$.*

Proof. The hypothesis means that there exist $w_1, ..., w_r \in W$ satisfying the following two conditions:

(i) The set of weights is root-distinct, i.e. $w_j\lambda - w_k\lambda \notin \Delta$.

(ii) $w_1\lambda, ..., w_r\lambda$ form a balanced simplex.

Thus the set $\{w_1\lambda, ..., w_r\lambda\}$ satisfies the hypothesis of Theorem 3.1. The construction from the proof of the theorem yields a generator f of the ring of invariants $\mathbb{C}[V(\lambda)]^G$, whose restriction to the secant space $\mathbb{P}\mathrm{Span}(v_{w_1\lambda}, ..., v_{w_r\lambda})$ is the monomial $(z_1^{b_1} \dots z_r^{b_r})^q$, where q is a positive integer, z_j are the coordinates associated to the weight vectors $v_{w_j\lambda}$, and $b_1, ..., b_r$ is the unique set of positive integers with greatest common divisor 1 such that $\sum b_j w_j \lambda = 0$. Now, clearly f does not vanish on $\mathbb{P}\mathrm{Span}(v_{w_1\lambda}, ..., v_{w_r\lambda}) \subset \sigma_r(\mathbb{X})$. $\square$

Example 3.2. *(Veronese varieties)*

Consider $G = SL_n$ acting on $V = S^k\mathbb{C}^n$ with $k, n \geq 2$. The associated homogeneous projective variety is the Veronese variety $\mathbb{X} = \mathrm{Ver}_k(\mathbb{P}^{n-1}) \subset \mathbb{P}(V)$. The extreme weight vectors for a given Cartan subgroup T the are k-th powers v_j^k of the corresponding basis vectors $v_1, ..., v_n \in \mathbb{C}^n$; the W-orbit of the highest weight $\lambda = k\omega_1$ forms a balanced $(n-1)$-simplex centered at 0, with $b_{W\lambda} = \#W\lambda = n$. This simplex is root-distinct, as $k \geq 2$. (For $k = 1$, the natural representation of SL_n has no root-distinct sets of weights with more than one element, which amounts to the fact that the projective space is a single SL_n-orbit.) Hence $\mathbb{C}[S^k\mathbb{C}^n]^{SL_n}$ admits a generator of degree qn for some $q \in \mathbb{N}$. It is not hard to see that, for $1 \leq r \leq n$, G has an open orbit in the secant variety Σ_r and the momentum image $\mu(\Sigma_r) \cap i\mathfrak{h}^*$ is the r-skeleton of the simplex $\mathrm{Conv}(W\lambda)$. Thus $\Sigma_r \subset \mathbb{P}^{us}$ for $r < n$ and $r_{ss} = n$. We can conclude that $d_1 \geq n$, where d_1 is the minimal degree of a nonconstant homogeneous polynomial in $\mathbb{C}[S^k\mathbb{C}^n]^{SL_n}$. For $k = 2$, the determinant of symmetric matrices is an invariant of degree n. The case $n = 3$ shows that, for large k, invariants in degree n may or may not occur.

Example 3.3. *Let us consider the case $\lambda = k\rho$, where $\rho = \frac{1}{2} \sum\limits_{\alpha \in \Delta^+} \alpha$. For $k \geq 2$, we have $\mu(\mathbb{P}) = \mu(\Sigma_2) = \mathrm{Conv}(Kk\rho)$.*

Remark 3.2. *The rank function* $\mathrm{rk}_{\mathbb{X}}$ *on* $\mathbb{P}$ *is independent of any group actions on* $\mathbb{X}$*, but the notions of rank of instability and semistability depend on the group. We use the notation* $r_{us,G}, r_{ss,G}$*, when the group needs to be specified. Certain homogeneous projective varieties admit transitive actions by proper subgroups of their automorphism group, say* $\tilde{G} \subset G = \mathrm{Aut}\mathbb{X}$*. Such transitive actions of subgroups have been classified by Onishchik. For simple* G*,* $\tilde{G}$ *is also simple, and the cases include* $\mathbb{P}^{2n-1}$*, homogeneous under* SL_{2n} *and* Sp_{2n}*; the varieties of pure spinors in the irreducible spin-representation, which are the same for* SO_{2n} *and* SO_{2n-1}*, the quadric* Q^5*, homogeneous under* SO_7 *and* G_2*. In general, a homogeneous projective variety has a semisimple automorphism group and splits as a product of homogeneous varieties* $\mathbb{X} = \mathbb{X}^1 \times \ldots \mathbb{X}^k$ *corresponding to the simple factors of* G*. Then a transitive subgroup* $\tilde{G} \subset G$ *is necessarily a product* $\tilde{G} = \tilde{G}^1 \times \cdots \times \tilde{G}^k$*, with* $\tilde{G}^j$ *transitive on* $\mathbb{X}^k$*.*

Let us consider the behaviour of the rank for (infinitesimally) simple G *and a proper subgroup* $\tilde{G}$ *acting transitively on* $\mathbb{X}$*. It turns out that* $\tilde{G}$ *is a simple group without outer automorphisms, i.e., all of its representations are self dual. Thus* $r_{us,\tilde{G}} = 1$*. Whenever semistable points exist (the only exception being the transitive action on the entire projective space* $\mathbb{X} = \mathbb{P}^{2n-1}$*, by* $\tilde{G} = Sp_{2n} \subset SL_{2n} = G$*, where we have* $r_{max} = 1$*) the rank of semistability for the subgroup is* $r_{ss,\tilde{G}} = 2$ *and the representation admits an* $\tilde{G}$*-invariant of even degree.*

Theorem 3.7. *Let* $\lambda \in \Lambda^+$*. Suppose that* $M \subset W$ *is a subset such that* $M\lambda$ *is a balanced simplex. Then, for some* $k \in \mathbb{N}$*, the representation* $V(k\lambda)$ *admits a generating* K*-invariant polynomial of degree* b_M*.*

Proof. First notice that for every $k \in \mathbb{N}$ we have $kM\lambda = Mk\lambda$. Furthermore, for $k \geq 2$, $kM\lambda$ is root-distinct. Thus, up to replacing λ by a multiple, we may assume that $M\lambda$ is root-distinct and hence, by Wildberger's lemma, $\mathbb{P}_{W\lambda} \cap \mathbb{P}(V(\lambda))^{ss} \neq \emptyset$.

Let $b_{w\lambda} \in \mathbb{N}$, for $w \in M$ be the integers satisfying $gcd\{b_{w\lambda} : w \in M\} = 1$ and $\sum\limits_{w \in M} b_{w\lambda} w\lambda = 0$. Put $b_{M\lambda} = \sum\limits_{w \in M} b_{w\lambda}$. Note that, for every $k \in \mathbb{N}$, one has

$$b_{w\lambda} = b_{wk\lambda} \quad , \quad b_{M\lambda} = b_{Mk\lambda} \ ,$$

so it makes sense to denote these numbers by b_w and b_M, respectively.

Let $v_{w\lambda} \subset V(\lambda)$ be a normalized (extreme) weight vector in $V(\lambda)$ with weight $w\lambda$ and let $z_{w\lambda}$ be the corresponding coordinate in $V(\lambda)^*$. Let $m_{M\lambda} = \prod\limits_{\nu \in M} z_{w\lambda}^{b_M} \in \mathbb{C}[V(\lambda)]_{b_M}$ be the monomial associated to $M\lambda$.

For every $k \in \mathbb{N}$, there is a natural embedding

$$V(k\lambda)^* \hookrightarrow \mathbb{C}[V(\lambda)]_k \ ,$$

which also yields an embedding

$$\mathbb{C}[V(k\lambda)]_l \hookrightarrow \mathbb{C}[V(\lambda)]_{kl} \ .$$

This we may identify $\mathbb{C}[V(k\lambda)]$ with a subring of $\mathbb{C}[V(\lambda)]$. We have

$$z_{wk\lambda} = z_{w\lambda}^k \in \mathrm{Ver}_k(Gz_\lambda) \subset V(k\lambda)^* \ .$$

Hence

$$m_{Mk\lambda} = m_{M\lambda}^k \in \mathbb{C}[V(k\lambda)]_{b_M} \subset \mathbb{C}[V(\lambda)]_{kb_M} .$$

By Theorem 3.1, for some $k \in \mathbb{N}$ there is a generator $f \in \mathbb{C}[V(\lambda)]^G_{kb_M}$ having the monomial $m^k_{M\lambda}$. The (G-equivariant) projection of f to $\mathbb{C}[V(k\lambda)]_{b_M}$, say f_1, is nonzero, having the monomial $m_{Mk\lambda}$. Hence

$$\mathbb{C}[V(k\lambda)]^G_{b_M} \neq 0 .$$

Furthermore, since $m_{Mk\lambda}$ cannot be decomposed as a nontrivial product of two other monomials on $V(k\lambda)$ of 0 weight, f_1 must be a generator. □

3.2 Rank-semi-continuous homogeneous projective varieties

In this section we consider a class of homogeneous projective varieties with particularly well-behaved secant varieties, defined as follows.

Definition 3.4. *A projective variety $\mathbb{X} \subset \mathbb{P}$ is called rank-semicontinuous, or rs-continuous, if the rank and border rank functions on $\mathbb{P}$ with respect to $\mathbb{X}$ coincide:* $\mathrm{rk}_{\mathbb{X}} = \underline{\mathrm{rk}}_{\mathbb{X}}$*, i.e., points of higher rank cannot be approximated and the Zariski closure in the definition of secant varieties is not necessary.*

Remark 3.3. *The problem of classifying rs-continuous homogeneous varieties was stated by Baur, Draisma and de Graaf in* [BDG07] *along with other open questions and conjectures concerning secant varieties of homogeneous varieties. In an earlier article,* [BD04]*, Baur and Draisma had studied adjoint varieties of simple classical groups, giving new proofs that these varieties are rs-continuous for SL_n and Sp_{2n}, while for SO_n they are not rs-continuous. In the latter case these authors also describe the nilpotent orbits appearing in the r-th secant variety. Landsberg and Manivel,* [LM03]*, studied the secant varieties of subcominuscule varieties (the closed projective orbits in the isotropy representations of irreducible Hermitian symmetric spaces); their results imply immediately rs-continuity for these varieties. The complete classification was obtained in [PT15] by A.V. Petukhov and the author and is stated below.*

Theorem 3.8. ([PT15]) *A homogeneous projective variety $\mathbb{X} \subset \mathbb{P}$ is rs-continuous if and only if $\Sigma_2 = \mathbb{X} \cup \mathbb{X}_2$, i.e., the required property holds for all r if and only if it holds for $r = 2$. The classification of the homogeneous rs-continuous varieties is given in the Table 1.*

Idea of proof: The proof is obtained in the following three steps.

1. We show that, if $V(\lambda)$ is an irreducible G-module such that $\mathbb{X} = G[v_\lambda]$ is rs-continuous, then λ is the sum of at most two (possibly) equal fundamental weights: $\lambda = \omega_i + \omega_j$. This is done using the fact that $\Sigma_2(\mathbb{X})$ is quasihomogeneous, i.e., $\Sigma_2(\mathbb{X}) = \overline{G[v_\lambda + v_{w_0\lambda}]}$, and contains the tangential variety of $\mathbb{X}$. We show that, if λ is the sum of three or more fundamental weights, then the tangential variety contains vectors of rank higher than 2.

2. We show that, if $\mathbb{X}$ is rs-continuous and $\lambda = \omega_i + \omega_j$, then G acts transitively on both $\mathbb{P}(V(\omega_i))$ and $\mathbb{P}(V(\omega_j))$. The only transitive actions on a projective space are given by the natural representations of $SL_{\ell+1}$ and $Sp_{2\ell}$. This yields the cases $\mathbb{X} = \mathbb{P}(\mathbb{C}^n)$, $\mathrm{Fl}_{1,n-1}(\mathbb{C}^n)$ and $\mathrm{Seg}(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1})$.

3. The situations where λ is a fundamental weight are considered case by case using the classification of simple groups.

Table 1

rs-continuous varieties, semistable range of rank, degrees of invariants

Variety $\mathbb{X} \subset \mathbb{P}(V)$	$G = \mathrm{Aut}\mathbb{X}$	$r_{ss}, ..., r_{max}$	$d_1, ..., d_k$
$\mathbb{P}(\mathbb{C}^n) = \mathbb{P}(\mathbb{C}^n)$	SL_n	$r_{us} = r_{max} = 1$	0
$\mathrm{Ver}_2(\mathbb{P}(\mathbb{C}^n)) \subset \mathbb{P}(\mathrm{S}^2\mathbb{C}^n)$	SL_n	$r_{ss} = r_{max} = n$	n
$\mathrm{Gr}_2(\mathbb{C}^n) \subset \mathbb{P}(\Lambda^2\mathbb{C}^n)$	SL_n	$r_{us} = r_{max} = \lfloor \frac{n}{2} \rfloor$ $r_{ss} = r_{max} = \lfloor \frac{n}{2} \rfloor$	0 for odd n; $n/2$ for even n
$\mathrm{Fl}_{1,n-1}(\mathbb{C}^n) \subset \mathbb{P}(\mathfrak{sl}_n)$	SL_n	$2, ..., n$	$2, ..., n$
$\mathrm{Q}^{n-2} \subset \mathbb{P}(\mathbb{C}^n)$	SO_n	2	2
$\mathrm{S}^{10} \subset \mathbb{P}^{15} = \mathbb{P}(\Lambda^{even}\mathbb{C}^5)$	Spin_{10}	$r_{us} = r_{max} = 2$	0
$\mathrm{Gr}_2(\mathbb{C}^{2n}, \omega) \subset \mathbb{P}(\Lambda_0^2\mathbb{C}^{2n})$	Sp_{2n}	2,...,n	2,4,6...,2n-2
$\mathrm{E}^{16} \subset \mathbb{P}^{26} = \mathbb{P}(\mathrm{Herm}_{3\times3}(\mathbb{O})_{\mathbb{C}})$	E_6	3	3
$\mathrm{F}^{15} \subset \mathbb{P}^{25} = \mathbb{P}(\mathrm{SHerm}_{3\times3}(\mathbb{O})_{\mathbb{C}})$	F_4	$2, 3$	$2, 3$
$\mathrm{Seg}(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}) \subset \mathbb{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$	$SL_m \times SL_n$	$r_{us} = r_{max} = \min\{m, n\}$ $r_{ss} = r_{max} = m$	0 for $m \neq n$; m for $m = n$

The notation in Table 1 is as follows: Ver_2 denotes the quadratic Veronese embedding; Gr_2 is the Grassmannian of 2-planes; $\mathrm{Fl}_{1,n-1}$ denotes the 2-step flag variety of lines contained in hyperplanes; Q^{n-2} is the $(n-2)$-dimensional quadratic hypersurface; S^{10} is the 10-dimensional variety of pure spinors; $\mathrm{Gr}_2(\mathbb{C}^{2n}, \omega)$ is the variety of 2-planes in $\mathbb{C}^{2n}$ isotropic for a given nondegenerate skew-symmetric form ω; $\mathrm{Herm}_{3\times3}(\mathbb{O})_{\mathbb{C}}$ denotes the complexified space of octonionic Hermitean 3×3-matrices and $\mathrm{SHerm}_{3\times3}(\mathbb{O})_{\mathbb{C}}$ is the subspace defined by vanishing of the trace; E^{16} is the set of such matrices with rank 1, which can be defined using the Freudenthal determinant and $\mathrm{F}^{15} = \mathrm{E}^{16} \cap \mathbb{P}(\mathrm{SHerm}_{3\times3}(\mathbb{O})_{\mathbb{C}})$; Seg denotes the Segre embedding of a product of projective spaces.

We obtain the following.

Theorem 3.9. *Suppose that $\mathbb{X} \subset \mathbb{P}$ is rs-continuous and G is the linear automorphism group of $\mathbb{X}$. Then the invariant ring is polynomial $\mathbb{C}[V(\lambda)]^G = \mathbb{C}[f_1, ..., f_k]$. The degrees of the generators, ordered nonincreasingly, coincide with the values of the rank function on the semistable locus:*

$$\{r_{ss}, ..., r_{max}\} = \{d_1, ..., d_k\}$$

except in the case of Sp_{2n} acting on $\mathrm{Gr}_2(\mathbb{C}^{2n}, \omega) \subset \mathbb{P}(\Lambda_0^2\mathbb{C}^{2n})$, where the two sets are related by the bijection $d_j = 2(r_j - 1)$. Furthermore, the generators can be chosen to vanish on the secant varieties as follows:

$$\Sigma_r \subset Z(f_{k-(r_{max}-r-1)}, ..., f_k), \ for \ \ r_{us} \leq r \leq r_{max} - 1.$$

The proof is obtained via case by case analysis using the classification of rs-continuous varieties presented above, the tables of [Kac80] containing the classification of representations whose invariant rings are isomorphic to polynomial rings, as well as some results on secant varieties of homogeneous projective varieties of small codimension from [Z93],§4.

4 Subgroup actions on flag varieties and GIT

In this section I present the results of my joint work with H. Seppänen, [ST18], concerning invariant theory and geometry of group actions related to embeddings of semisimple complex algebraic groups, and discussed in section 1.2 of the Introduction.

4.1 Setting and statement of the main results

Let $\hat{G} \subset G$ be an embedding of connected complex semisimple algebraic groups. We also assume that G is simply connected. Let $X = G/B$ be the complete flag variety of G. Recall that for $\lambda \in \Lambda^+$, $\mathcal{R}_\lambda$ denotes the ring of sections of the line bundle $\mathcal{L}_\lambda$ on X:

$$\mathcal{R}_\lambda = \bigoplus_{j=0}^{\infty} H^0(X, \mathcal{L}_\lambda^j) = \bigoplus_{j=0}^{\infty} V(j\lambda)^* .$$

The basic GIT notions relating the geometry of $X = G/B$ to the invariant rings $R_\lambda^{\hat{G}}$ are the notions of instability, semistability, stability and quotients, introduced in section 2. For us here, the central role is played by the $\hat{G}$-unstable locus, defined by the vanishing of the nonconstant invariants in the section ring of a given ample line bundle. The ample line bundles on X have the form $\mathcal{L}_\lambda$ for $\lambda \in \Lambda^+$, and one has

$$X^{us}(\lambda) = X^{us}_{\hat{G}}(\lambda) = Z(J_\lambda) \subset X \ , \quad J_\lambda = \bigoplus_{j\geq 1}^{\infty} H^0(X, \mathcal{L}_\lambda^j)^{\hat{G}} = \bigoplus_{j\geq 1}^{\infty} (V(j\lambda)^*)^{\hat{G}} .$$

The semistable locus is the complementary open set $X^{ss}(\lambda) = X^{ss}_{\hat{G}}(\lambda) = X \setminus X^{us}(\lambda)$. The Hilbert-Mumford criterion (see section 2) gives a numerical characterization of the unstable or, equivalently, semistable points, and allows to extend the notions to the $\mathbb{R}$-Picard group, i.e., $\lambda \in \Lambda_{\mathbb{R}}$. This yields a characterization of the $\hat{G}$-ample line bundles by having an unstable locus of positive codimension or, equivalently, a nonempty semistable locus. *In some sense, the simple choice, whether to focus on unstable or semistable points, represents the main difference between our view on $C^{\hat{G}}(X)$ and the view we see in the articles mentioned above.* The cohomological conditions for w are obtained from the condition for its Schubert cell to contain semistable points. We focus on instability.

For the Littlewood-Richardson monoid LR_0 we need to consider nonregular dominant weights $\lambda \in \Lambda^+$ as well. These yield semiample line bundles and the above definition of the unstable locus extends to that case. However, the structure of quotients may differ. We define the $\hat{G}$-ample cone on X as the closed cone generated by ample line bundles as

$$C^{\hat{G}}(X) = \overline{\{\lambda \in \Lambda_{\mathbb{Q}}^{++} : \exists q \in \mathbb{N} : J_{q\lambda} \neq 0\}} \subset \Lambda_{\mathbb{R}} \ .$$

Whenever $C^{\hat{G}}(X)$ is nonempty, we have

$$C^{\hat{G}}(X) = \mathcal{LR}_0(\hat{G} \subset G) = \{\lambda \in \Lambda_{\mathbb{R}}^+ : \mathrm{codim}_X X^{us}_{\hat{G}}(\lambda) > 0\} \subset \Lambda_{\mathbb{R}} \ .$$

A typical case where $C^{\hat{G}}(X) = \emptyset$ is given by the condition that $\hat{G}$ and G have nontrivial common connected normal subgroups. This condition is in fact necessary and sufficient for the full branching cone $\mathcal{LR}$ to have full dimension in $\hat{\Lambda}_{\mathbb{R}} \times \Lambda_{\mathbb{R}}$, cf. [R10]. The cone $\mathcal{LR}_0$ may have positive codimension in $\Lambda_{\mathbb{R}}$ and contain, or not contain, regular weights. For instance, for $\hat{G} = Sp_{2n} \subset SL_{2n} = G$ we have $C^{\hat{G}}(X) = \emptyset$, while for $\hat{G} \overset{diag}{\hookrightarrow} \hat{G} \times \hat{G} = G$, we have $C^{\hat{G}}(X) = \mathcal{LR}_0 = \{(\hat{\lambda}, \hat{\lambda}^*) : \hat{\lambda} \in \hat{\Lambda}^+\}$.

Our first main result is an explicit description of the $\hat{G}$-unstable locus in $X = G/B$ with respect to any ample line bundle, stated as Theorem I below. The formula for the unstable locus is proven in Theorem 4.4 and the dimension formula is proven in Theorem 4.7 which describes the Kirwan-Ness stratification. The formula stated below contains some redundancies, which are removed in the referred theorems at the expense of more technically involved statements.

Recall that the T-fixed points in X are parametrized by the elements of the Weyl group W, as $X^T = \{x_w = wB, w \in W\}$; their B-orbits give the Schubert cell decomposition $X = \sqcup_w B x_w$. Suppose, which can be done without loss of generality, that Weyl chambers for $\hat{G}$ and G are chosen so that $\dim \mathfrak{t}_+ \cap \hat{\mathfrak{t}}_+ = \dim \hat{\mathfrak{t}}_+$ (real dimension). For any one-parameter subgroup of T, $\xi \in \Gamma$, let $P_\xi \subset G$ denote the parabolic subgroup, whose Lie algebra is the sum of the eigenspaces of $\mathrm{ad}\xi$ with nonnegative eigenvalues. Let $P_1, ..., P_q$ be the maximal parabolic subgroups of G among P_ξ with $\xi \in \hat{\Gamma}^+ \setminus \{0\}$. Then there are uniquely determined indivisible dominant one-parameter subgroups $\xi_1, ..., \xi_q \in \hat{\Gamma}^+$ such that $P_j = P_{\xi_j}$. The set $\xi_1, ..., \xi_q$ is known to be been related to the Littlewood-Richardson cone, notably in the works of Ressayre, [R10]. We show how it arises naturally in the study of the Kirwan-Ness stratification. For diagonal embeddings the ξ_j's are just the fundamental coweights of $\hat{G}$. Set $r_j = \dim G/P_j$, $\hat{r}_j = \dim \hat{G}/\hat{P}_j$.

Theorem I: *Let $\lambda \in \Lambda^{++}$. Then the $\hat{G}$-unstable locus can be written as the $\hat{G}$-saturation of a union of parabolic orbits*

$$X^{us}(\lambda) = \bigcup_{j=1}^{q} \bigcup_{w \in W : \lambda(w^{-1}\xi_j) > 0} \hat{G} P_j x_w \ .$$

Furthermore, denoting $p_j(w) = \dim P_j x_w$, we have

$$\dim \hat{G} P_j x_w \leq \hat{r}_j + p_j(w)$$
$$\dim X^{us}(\lambda) = \max_{j,w} \{\hat{r}_j + p_j(w) \ : \ \lambda(w^{-1}\xi_j) > 0, \ \dim \hat{G} P_\xi x_w = \hat{r}_j + p_j(w)\} \ .$$

The closure of a parabolic orbit $\overline{P_\xi x_w}$ is a Schubert variety, perhaps not for B, but for a Borel subgroup $B^\xi \subset P_\xi$, Weyl-conjugate to B, for which ξ is dominant in G. The dimension and codimension of $P_\xi x_w$ can be computed in terms of lengths of Weyl group elements, which is very useful for our calculations. The dimension of the saturation $\hat{G} P_\xi x_w$ is a more delicate issue; in general, the inequality given in the above theorem may be strict. It turns out, however, due to the general dimension formulae for Kirwan-Ness stratifications, that it suffices to consider only pairs (ξ, w) for which the equality $\dim \hat{G} P_\xi x_w = \hat{r}_\xi + p_\xi(w)$ holds. We call such pairs fit (in the formal Definition 4.3, we also we make some additional technical requirement reducing redundancies in the choice of ξ and w defining the same parabolic orbit). We prove a combinatorial property of fit pairs allowing us to study the variation of the dimension of the unstable locus along variations of λ, as explained below.

The condition $\dim \hat{G} P_j x_w = \hat{r}_j + p_j(w)$ is likely related to the notions of Levi-movability and the Belkale-Kumar product cohomology of flag varieties, a key notion in the minimal description of the $\hat{G}$-ample cone in the works of Belkale-Kumar, Ressayre, Richmond, [BK06],[R10],[RR11]. We develop an independent approach focused on the unstable locus, based directly on the Hilbert-Mumford criterion and the Kirwan-Ness stratification theorem. Our method is rather related to a method used by Popov, [P03], to study unstable loci representation spaces. Also, as an addendum given in section 4.5 which is independent from the rest of our results, we adopt a combinatorial tree-algorithm from [P03], which can be used to determine the set of relevant (stratifying) pairs (j, w).

As a corollary, we obtain a cohomology-free description of $C^{\hat{G}}(X)$, where the cohomological condition is replaced by the dimension condition $\dim \hat{G} P_j x_w = \hat{r_j} + p_j(w)$ for concrete Schubert varieties. Our description is not necessarily optimal, redundant inequalities may occur. It is, however, exact and allows us to study the interior of the $\hat{G}$-ample cone.

Example 4.1. *(Three dimensional simple subgroups)*

Suppose that $\mathrm{rank}(\hat{G}) = 1$*, so that* $\hat{G}$ *is infinitesimally isomorphic to* SL_2*. Then* $\hat{\Gamma}^+$ *is generated by a unique indivisible element,* $\xi = \xi_1$*. Let* $B \subset G$ *be a Borel subgroup for which* ξ *is dominant and let* $l(w) = \dim B x_w$ *denote the length of Weyl group element. It is clear that, for all* $w \neq w_0$ *(the longest element of* W*), we have* $\dim \hat{G} P_\xi x_w = 1 + p_\xi(w)$*. The unstable locus for* $\lambda \in \Lambda^{++}$*, and its codimension are given by*

$$X^{us}(\lambda) = \bigcup_{w \in W : \lambda(w^{-1}\xi) > 0} \hat{G} P_\xi x_w \ ,$$

$$\mathrm{codim}_X X^{us}(\lambda) = -1 + \min\{l(w_0 w) : w \in W, \lambda(w^{-1}\xi) > 0\} \ .$$

Consequently, the $\hat{G}$*-ample cone is given by*

$$C^{\hat{G}}(X) = \{\lambda \in \Lambda^+_{\mathbb{R}} \ : \ \lambda(w^{-1}\xi) < 0 \ , \ \forall w \in W : l(w) = 1\} \ .$$

The $\hat{G}$-ample cone is subdivided into GIT-equivalence classes, defined by equality of the unstable loci, i.e., $\lambda \sim \lambda'$ if and only if $X^{us}(\lambda) = X^{us}(\lambda')$. For $\hat{G}$-ample line bundles, the projective spectrum of the invariant ring is isomorphic to the GIT-quotient of X defined by Hilbert's equivalence relation on the semistable locus:

$$Y_\lambda = X^{ss}(\lambda)//\hat{G} \cong \mathrm{Proj}(R_\lambda^{\hat{G}}) \quad , \quad x_1 \sim x_2 \Longleftrightarrow \overline{\hat{G}x_1} \cap \overline{\hat{G}x_2} \cap X^{ss}(\lambda) \neq \emptyset \ .$$

The quotients defined by GIT-equivalent bundles are clearly isomorphic, and we denote $X^{us}(C) = X^{us}(\lambda)$ and $Y_C = Y_\lambda$ for a GIT-class $C \ni \lambda$. The GIT-classes form a fan of cones in $C^{\hat{G}}(X)$, and there are rational maps between some of these quotients, depending on relations between the corresponding GIT-classes (cf. [DH98], [T96], [R00]).

Some important properties of the quotient are reflected in properties of the unstable locus. Note that the quotient is geometric when the semistable orbits are equidimensional. In particular, one considers the set of infinitesimally free orbits closed in the semistable locus, called the stable locus:

$$X^s(\lambda) = X^s_{\hat{G}}(\lambda) = \{x \in X^{ss}_{\hat{G}}(\lambda) : \hat{G}x \subset X^{ss}_{\hat{G}}(\lambda) \text{ closed}, \dim \hat{G}_x = 0\} \ ,$$

where $\hat{G}_x$ denotes the stabilizer of x. The GIT-class of λ is called a chamber if all semistable points are stable, i.e., $X^{us}_{\hat{G}}(\lambda)$ contains all points with positive dimensional stabilizer. It is shown in [S14] that, for embeddings of semisimple groups acting on complete flag varieties, the chambers are exactly the full-dimensional GIT-classes in $C^{\hat{G}}(X)$. The following theorem is perhaps known to experts, but we state it here since it is important in our setting, and we present a proof in the text (see Theorem 4.9).

Theorem II: *The $\hat{T}$-ample cone on X consists of the entire Weyl chamber and $\hat{T}$-chamber structure is defined by the hyperplanes $\mathcal{H}_{w\xi_j}$ orthogonal to $w\xi_j$ for $w \in W$ and $j = 1, ..., q$.*

The $\hat{G}$-chambers, whenever they exist, are convex cones, open in $\Lambda_{\mathbb{R}}$, spanned by certain unions of $\hat{T}$-chambers. In particular, all hyperplanes bounding $\hat{G}$-chambers are of the form $\mathcal{H}_{w\xi_j}$.

The Picard group of the quotient Y_λ is naturally related to the Picard group of X, cf. [KKV89]. The relation becomes simpler when the unstable locus does not contain divisors. This motivates the definition of $\hat{G}$-movable GIT-classes as those whose unstable locus has codimension at least 2. The union of these classes forms a cone, called the $\hat{G}$-movable cone on X, denoted by

$$\mathrm{Mov}^{\hat{G}}(X) = \overline{\{\lambda \in \Lambda^{++}_{\mathbb{Q}} : \mathrm{codim}_X X^{us}_{\hat{G}}(\lambda) \geq 2\}} \subset C^{\hat{G}}(X) \ .$$

A *$\hat{G}$-movable chamber* is a full-dimensional GIT-class C satisfying $X^{ss}(C) = X^s(C)$ and $\mathrm{codim}_X X^{us}(C) \geq 2$. In such a case, we obtain a geometric quotient Y_λ whose Picard group embeds, via pullback followed by extension, as a sublattice of full rank in the Picard group of X, yielding an isomorphism over the reals.

Such a quotient is shown in [S14] to be a Mori dream space whose effective cone is identified with $C^{\hat{G}}(X)$. **The question** arises: do $\hat{G}$-movable chambers exist, or under what conditions?

The requested chambers are not always present. An important class of counterexamples is supplied by spherical subgroups $\hat{G} \subset G$, where $\dim V(\lambda)^{\hat{G}} \leq 1$ for all λ: there are no $\hat{G}$-movable line bundles and the quotient is a point. There are also non-spherical cases, like $SL_2 \subset SL_2^{\times 4}$, where the $\hat{G}$-movable cone is the diagonal ray $(\mathbb{R}_+)\rho$. In our previous work [ST15] we have obtained detailed results for $\hat{G}$ a principal SL_2-subgroup of a semisimple group G; in this case $\hat{G}$-movable chambers exist if $\dim X \geq 5$, which for simple G means not to be of type A_2 or B_2.

Using our formula for the unstable locus, we obtain a concrete description of a system of nested cones in $\Lambda^+_{\mathbb{R}}$ defined by codimension of the unstable locus, beginning with the $\hat{G}$-ample and the $\hat{G}$-movable cones.

Theorem III: *The sets* $C^{\hat{G}}_k(X) = \overline{\{\lambda \in \Lambda^{++}_{\mathbb{Q}} : \mathrm{codim}_X X^{us}(\lambda) \geq k\}} \subset \Lambda^+_{\mathbb{R}}$, *defined for* $k \geq 1$, *form a finite sequence of nested rational polyhedral cones in* $\Lambda^+_{\mathbb{R}}$. *Whenever* $C^{\hat{G}}_k(X)$ *is nonempty, it is given by*

$$C^{\hat{G}}_k(X) = \{\lambda \in \Lambda^+_{\mathbb{R}} : \lambda(w^{-1}\xi_j) \leq 0, \forall j, w : \dim \hat{G}P_j x_w = \hat{r}_j + p_j(w) = \dim X - k + 1\}.$$

Moreover, the cone $C^{\hat{G}}_{k+1}(X)$ *is contained in the interior of* $C^{\hat{G}}_k(X)$, *in the topology of the Weyl chamber.*

The $\hat{G}$*-ample and -movable cones are obtained for* $k = 1$ *and* 2, *respectively.*

The proof is given in Theorem 4.13. Note that it is not a priori clear that the codimension of the unstable loci could not make "jumps" and increase in steps bigger than one when passing from one GIT class in $C^{\hat{G}}(X)$ to another. The following "no jumps" result (cf. Lemma 4.14) is a key step in our proof of the above theorem, and presents an interest by itself.

No jump lemma: *Suppose that* $C_1, C_2 \subset C^{\hat{G}}(X)$ *are GIT-classes in the* $\hat{G}$*-ample cone satisfying* $\overline{C_1} \supset C_2$. *Then*

$$|\mathrm{codim}_X X^{us}(C_1) - \mathrm{codim}_X X^{us}(C_2)| \leq 1 \ .$$

The same inequality holds if C_1, C_2 *are GIT-chambers sharing a facet.*

The above theorem has the following direct corollary.

Corollary: (i) *The* $\hat{G}$*-ample cone is nonempty if and only if* $\hat{G}$ *does not act transitively on any partial flag variety of* G. *Whenever this is the case, we have*

$$C^{\hat{G}}(X) = \{\lambda \in \Lambda^+_{\mathbb{R}} : \ \lambda(w^{-1}\xi_j) \leq 0 \ \ , \forall j, w : \hat{G}\overline{P_j x_w} = X\}.$$

For line bundles on its regular boundary, $\lambda \in \Lambda^{++} \cap \partial C^{\hat{G}}(X)$, *one has unstable locus of codimension 1.*

(ii) *$\hat{G}$-movable chambers exist if and only if the cone $C_2^{\hat{G}}(X)$ has full dimension.*

(iii) *If $C_3^{\hat{G}}(X) \neq \emptyset$, then $\hat{G}$-movable chambers exist.*

Concerning a more easily computable criterion, we derive the following numerical sufficient condition for presence of $\hat{G}$-movable chambers, expressed in terms of some structure constants of the embedding $\hat{G} \subset G$. It is obtained by considering the anticanonical bundle on X, i.e., $\lambda = 2\rho$, the sum of the positive roots of G, which tends, heuristically, to have a small unstable locus. The proof is given in Propositions 4.16 and 4.17.

Corollary: *For $j = 1, ..., q$, let a_j and b_j denote, respectively, the minimal and maximal positive value of a root of G on ξ_j. If $\min_j\{\frac{a_j}{a_j+b_j} r_j - \hat{r}_j\} \geq 2$, then X admits $\hat{G}$-movable chambers.*

In particular, $\hat{G}$-movable chambers exist for diagonal embeddings $\hat{G} \subset G = \hat{G}^{\times k}$ with sufficiently large k. If $\hat{G}$ is a product of classical groups, it suffices to take $k \geq 5$.

The next theorem concerns the quotients arising from $\hat{G}$-movable chambers, their Picard groups and Cox rings. Refining the aforementioned results [S14] on the effective cone on the quotient, we find a natural identification between the GIT-equivalence relation in $\mathrm{Pic}(X)$ with the Mori equivalence relation in $\mathrm{Pic}(Y)$, see also [HK00]. I refer to our article [ST18] relevant definitions and the proof of this theorem.

Theorem IV: *Suppose that there exists a $\hat{G}$-movable chamber $C \subset C^{\hat{G}}(X)$ and let $Y = Y_C$ be the corresponding GIT-quotient of X. Then Y is a Mori dream space and there is a canonical isomorphism of $\mathbb{R}$-Picard groups giving rise to the following identifications:*

$$\begin{array}{rcl}
\mathrm{Pic}(X)_{\mathbb{R}} & \cong & \mathrm{Pic}(Y)_{\mathbb{R}} \\
C^{G}(X) & \cong & \overline{\mathrm{Eff}}(Y) \\
\textit{GIT-chambers} & \leftrightarrow & \textit{Mori chambers} \\
\mathrm{Mov}^{\hat{G}}(X) & \cong & \mathrm{Mov}(Y) \\
\overline{C} & \cong & \mathrm{Nef}(Y) \\
\mathrm{Cox}(X)^{\hat{G}} \cong & \bigoplus_{\lambda \in \Lambda^+} V(\lambda)^{\hat{G}} & \cong \textit{ Finite extension of } \mathrm{Cox}(Y)\,.
\end{array}$$

Moreover, all rational contractions of Y to normal projective varieties are induced by VGIT from X.

Let us note that the family of Mori dream spaces produced as GIT-quotients of flag varieties could be of independent interest. For the sake of representation theory, clearly a concrete model for Y would be of great benefit – as explained above, this variety would encode the full information on dimensions of $\hat{G}$-invariants in G-modules for the given subgroup $\hat{G} \subset G$. Although we are able to prove many nice properties, these spaces remain somewhat implicit, as is often the case with quotients, due to the implicit nature of the fundamental existence results in invariant theory. The same is true to some extent for Mori dream spaces since several general constructions involve quotients, while many explicit alterations of varieties destroy the Mori dream property. It is therefore of interest to know whether our quotients appear among the known examples of Mori dream spaces. Perhaps the interaction of the Mori theory with the structure theory of semisimple groups could help to obtain more concrete information about this family of spaces, ideally build concrete models at least for special classes of subgroups like diagonals.

4.2 Instability on flag varieties and Schubert varieties

In this section we prove our formula for the unstable locus presented as Theorem I in the previous section. The key observation is that, when $X = G/B$, the connected components of the Hesselink blades are not only preserved by the parabolic subgroups of $\hat{G}$, but are in fact orbits of parabolic subgroups of G through T-fixed points, in a way that $X_{\xi,m} = \cup P_\xi x_w$, union over w such that $w\lambda(\xi) = m$. We begin by summarizing some facts about instability on flag varieties, which will also help introduce relevant objects and notation.

4.2.1 One-parameter subgroups

For a semisimple element $\xi \in \mathfrak{g}$ let $\mathfrak{l}_\xi$ denote the 0-eigenspace of $\mathrm{ad}\xi$, i.e., the centralizer of ξ, let $\mathfrak{p}_\xi$ denote sum of the nonnegative eigenspaces, and let $\mathfrak{r}_\xi^\pm$ denote the sum of the positive/negative eigenspaces. Then $\mathfrak{p}_\xi$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi decomposition $\mathfrak{p}_\xi = \mathfrak{l}_\xi \oplus \mathfrak{r}_\xi^+$. We denote the corresponding subgroups of G by $P_\xi, L_\xi, R_\xi^\pm$ and also write $R_\xi = R_\xi^+$. Recall that an element $\xi \in \mathfrak{g}$ is called regular if its centralizer is a Cartan subalgebra or, equivalently, $\mathfrak{p}_\xi$ is a Borel subalgebra. We denote by $\mathfrak{g}_{\mathrm{reg}}$ the set of regular semisimple elements, and for any subset $A \subset \mathfrak{g}$ we denote $A_{\mathrm{reg}} = A \cap \mathfrak{g}_{\mathrm{reg}}$. An element of our fixed Cartan subalgebra $\mathfrak{t}$ is regular if it belongs to the interior of some Weyl chamber, i.e., no root in $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ vanishes on it.

Lemma 4.1. *Let $\xi \in \Gamma^+$ be a dominant OPS of G with respect to a Borel subgroup B. Let $P_\xi = L_\xi R_\xi$ be the associated parabolic subgroup. Then the following hold:*

(i) *The set of fixed points of ξ in X consists of the union of the closed L_ξ-orbits, which are exactly the L_ξ-orbits of the T-fixed points, or the L_ξ-orbits of the $B \cap L_\xi$-fixed points, parametrized by the left coset space $W_\xi \setminus W$, or by the set ${}^\xi W$ of shortest representatives:*

$$X^\xi = \bigcup_{w \in W} L_\xi x_w = \bigsqcup_{w \in {}^\xi W} L_\xi x_w \ .$$

(ii) *Every P_ξ-orbit in X contains exactly one closed L_ξ-orbit. Every P_ξ-orbit contains a unique open B-orbit and a unique B-orbit of minimal dimension.*

These correspond to a pair of elements $w^1, w_1 \in W_\xi w$ in every coset, having, respectively, maximal and minimal length with respect to B related by $w^1 = w_{01} w_1$, where w_{01} is the longest element in W_ξ with respect to $B_\xi = B \cap L_\xi$. The closure of every P_ξ-orbit is a Schubert variety $\overline{P_\xi x_w} = \overline{B x_{w^1}}$. The dimension and codimension of an orbit are given by

$$\dim P_\xi x = l(w^1) = n_\xi + l(w_1) \quad , \quad \operatorname{codim}_X P_\xi x = r_\xi - l(w_1) \ .$$

where $n_\xi = \dim P_\xi / B$ and $r_\xi = \dim R_\xi$.

(iii) *For $\lambda \in \Lambda^+$, the Mumford function M^ξ defined in (2) is constant on P_ξ-orbits and its values are given by the weights corresponding to the T-fixed points:*

$$M^\xi(x) = M^\xi(x_w) = w\lambda(\xi) \quad \text{for} \quad x \in P_\xi x_w \ .$$

The ξ-unstable locus in X with respect to λ is given by

$$X_\xi^{us}(\lambda) = \bigsqcup_{w \in {}^\xi W^+(\lambda,\xi)} P_\xi x_w = \bigcup_{w \in W^+(\lambda,\xi)_{B-\max}} \overline{B x_w} \ ,$$

where $W^+(\lambda,\xi) = \{w \in W : w\lambda(\xi) > 0\}$. The first union is disjoint, while the second one gives exactly the irreducible components.

Proof. For (i), since L_ξ acts on X^ξ and $X^T \subset X^\xi$ we have $L_\xi x_w \subset X^\xi$ for all w. On the other hand, we note that any fixed point $x \in X^\xi$ belongs to a unique Schubert cell $B x_w$. Using the fact that $B x_w = N x_w$ and the global linearization of the T-action on $N x_w$, one observes that $X^\xi \cap N x_w = N_\xi x_w \subset L_\xi x_w$.

For (ii), note first that the B-orbits in a given P_ξ-orbit at the orbits through its T-fixed points. By irreducibility of orbit-closures every P_ξ-orbit contains a unique open B-orbit, say $\overline{B x_{w^1}} = \overline{P_\xi x_{w^1}}$. On the other hand, computing the tangent spaces in terms of roots, one sees that for any $w \in W$, $B x_w$ is open in $P_\xi x_w$ if and only if N_ξ^- fixes x_w. There is a unique such point in every closed L_ξ-orbit, hence there is a unique closed L_ξ-orbit in every P_ξ-orbit, and the T-fixed points in $P_\xi x_w$ form a single W_ξ-orbit. By its definition w^1 has maximal length in $W_\xi w^1$, equal to the dimension of $P_\xi w^1$. One has $l(ww^1) = l(w^1) - l(w)$ for $w \in W_\xi$. The longest element $w_{01} \in W_\xi$ defines $w_1 = w_{01} w^1$ of length $l(w_1) = l(w^1) - n_\xi$. This yield the dimension formulae.

For (iii), recall that the embedding $\phi_\lambda : X \hookrightarrow \mathbb{P}(V(\lambda))$ defined by the line bundle $\mathcal{L}_\lambda$ sends x_w to the extreme weight vector $[v_{w\lambda}]$. The Hilbert-Mumford criterion brings us to consider the following partition of the Weyl group (determined for any pair $(\lambda, \xi) \in \Lambda \times \Gamma$):

$$\begin{aligned} W = W^+(\lambda,\xi) \sqcup W^0(\lambda,\xi) \sqcup W^-(\lambda,\xi) \quad , \\ W^+(\lambda,\xi) = \{w \in W : w\lambda(\xi) > 0\} \ , \\ W^0(\lambda,\xi) = \{w \in W : w\lambda(\xi) = 0\} \ , \\ W^-(\lambda,\xi) = \{w \in W : w\lambda(\xi) < 0\} \ . \end{aligned} \tag{5}$$

If $w, w' \in W$ are related by the Bruhat order as $w' \leq w$, then $w'\lambda(\xi) \geq w\lambda(\xi)$ for all $\xi \in \mathfrak{t}_+$. Indeed, The Bruhat order is defined by $w' \leq w$ if $x_{w'} \in \overline{B x_w} \subset X$, and $w' < w$ holds if $w' \neq w$. The linear span of the Schubert variety in $V(\lambda)$ is the Demazure B-module $V_{B,w\lambda}$ whose weights are exactly the weights of $V(\lambda)$ contained in $w\lambda + Q_+$. Thus $w'\lambda = w\lambda + q$ for some sum of positive roots q. If $h \in i\mathfrak{t}_+$, then $q(h) \geq 0$ and hence $w'\lambda(h) \geq w\lambda(h)$.

Consequently, if w belongs to either W^+ or $W^+ \cup W^0$, then so do all elements smaller than w. Hence it suffices to take Bruhat-maximal elements as indices for the union. □

4.2.2 Compatible Weyl chambers and cubicles

Returning to our embedding $\hat{G} \subset G$, we have now two notions of regularity on $\hat{\mathfrak{g}}$, its intrinsic one, and the one induced by the embedding in $\mathfrak{g}$. We shall use the subscript reg only for the G-notion and use $\hat{G}$-reg for the intrinsic notion on $\hat{\mathfrak{g}}$ or whenever more precision is necessary. So, for instance, $\hat{\Gamma}^+_{\mathrm{reg}}$ denotes the set of $\hat{B}$-dominant, G-regular OPS of $\hat{T}$. Clearly G-regular implies $\hat{G}$-regular, but in general the converse implication does not hold. It holds if and only if any Weyl chamber $\hat{\mathfrak{t}}_+$ is contained in a unique Weyl chamber $\mathfrak{t}_+$,

The calculation of the Mumford function in Lemma 4.1, (iii), concerns a weight λ and an OPS ξ dominant with respect to the same Weyl chamber in $\mathfrak{t}$. However, for the $\hat{G}$-unstable loci of the line bundles given by λ in a given Λ^+, we need to handle OPS from $\hat{\Gamma}^+$, which is not necessarily contained in Γ^+. To this end we follow Berenstein and Sjamaar, [BS00], who introduced the following notions associated in general to a pair $\hat{\mathfrak{g}} \subset \mathfrak{g}$ of reductive complex Lie algebras with a fixed pair $\hat{\mathfrak{t}} \subset \mathfrak{t}$ of nested Cartan subalgebras. Two Weyl chambers $\hat{\mathfrak{t}}_+$ and $\mathfrak{t}_+$ are called *compatible* if $\dim_{\mathbb{R}} \hat{\mathfrak{t}}_+ \cap \mathfrak{t}_+ = \dim_{\mathbb{C}} \hat{\mathfrak{t}}$. Let us fix from now on a compatible pair of chambers $\hat{\mathfrak{t}}_+$ and $\mathfrak{t}_+$. The Weyl chambers of $\mathfrak{t}$ are parametrized by Weyl group elements, and the chambers compatible with $\hat{\mathfrak{t}}_+$ determine the following set

$$W_{\mathrm{com}} = \{w \in W : \dim_{\mathbb{R}}(\hat{\mathfrak{t}}_+ \cap w\mathfrak{t}_+) = \dim_C \hat{\mathfrak{t}}\} ,$$

called *the compatible Weyl set.* For $\sigma \in W_{\mathrm{com}}$, the cone

$$\hat{\mathfrak{t}}_\sigma = \hat{\mathfrak{t}}_+ \cap \sigma\mathfrak{t}_+$$

is called a *cubicle* in $\hat{\mathfrak{t}}$. We have

$$\hat{\mathfrak{t}}_+ = \bigcup_{\sigma \in W_{\mathrm{com}}} \hat{\mathfrak{t}}_\sigma .$$

Berenstein and Sjamaar observed that the Weyl group of the centralizer $Z_G(\hat{T})$ acts on W_{com}, and defined *the relative Weyl set* W_{rel} to be the set of shortest representatives of the respective coset space.

Proposition 4.2. *For every nonzero $\xi \in \hat{\Gamma}^+$ there exists a unique element in W_{rel}, to be denoted by σ_ξ, such that $\sigma_\xi^{-1}\xi \in \Gamma^+$.*

Remark 4.1. *Under the hypothesis that $\hat{G}$ contains regular elements of G, the centralizer $Z_G(\hat{T})$ equals the Cartan subgroup T of G and has trivial Weyl group*

$W_T(T) = \{1\}$. *Let* $\mathfrak{t}_+^\circ$ *denote the relative interior of the Weyl chamber* $\mathfrak{t}_+$. *The compatible Weyl chambers are exactly those whose relative interiors intersect* $\hat{\mathfrak{t}}_+^\circ$. *Hence we have*

$$W_{\mathrm{com}} = W_{\mathrm{rel}} = \{w \in W : \hat{\mathfrak{t}}_+^\circ \cap w\mathfrak{t}_+^\circ \neq \emptyset\} \ .$$

Note that the Borel subgroups B^σ *for* $\sigma \in W_{\mathrm{com}}$ *contain the fixed* $\hat{B}$, *but they might not be all Borel subgroups of* G *containing* $\hat{B}$. *More occur, for instance, for a root-subgroup* $SL_2 \subset SL_3$.

Since $T = Z_G(\hat{T}) = Z_G(T)$ and $Z_G(\hat{T}) \subset N_G(\hat{T})$ is a normal subgroup, we have $N_{\hat{G}}(\hat{T}) \subset N_G(\hat{T}) \subset N_G(T)$. This yields an inclusion

$$\mathrm{j} : \hat{W} \subset W \ .$$

The inclusion $\hat{T} \subset T$ is equivariant with respect to j. There is a j-duality involution on W given by $w \mapsto w^* = \mathrm{j}(\hat{w}_0)ww_0$. Lemma 2.4.3. in [BS00] states that W_{com} is stable under j-duality, and the cubicles are permuted by $\hat{\mathfrak{t}}_{\sigma^*} = -\hat{w}_0\hat{\mathfrak{t}}_\sigma$ for $\sigma \in W_{\mathrm{com}}$.

Let $\sigma \in W_{\mathrm{com}}$ and $\lambda \in \Lambda^{++}$. Then $\iota^*(\sigma\lambda) \in \tilde{\Lambda}^+$ and hence $\sigma \in W^+(\lambda, \xi)$ and $\mathrm{j}(\hat{w}_0)\sigma \in W^-(\lambda, \xi)$ for any $\xi \in \hat{\mathfrak{t}}_+^\circ$. If $\xi \in \hat{\mathfrak{t}}_\sigma$, then $\sigma\lambda(\xi)$ is the maximum of $W\lambda(\xi)$.

Definition 4.1. *For any* $\xi \in \hat{\Gamma}^+ \setminus \{0\}$ *we fix a relative Weyl group element* $\sigma_\xi \in W_{\mathrm{rel}}$ *such that* ξ *belongs to the cubicle* $\hat{\mathfrak{t}}_{\sigma_\xi}$. *We denote* $B_\xi = L_\xi \cap B^{\sigma_\xi}$, *this is a Borel subgroup of* L_ξ *such that* $\hat{B}_\xi = B_\xi \cap \hat{L}_\xi$ *is a Borel subgroup of* $\hat{L}_\xi$. *We denote by* $l_\xi(w) = \dim B^{\sigma_\xi} x_{w\sigma_\xi}$ *the* B^{σ_ξ}*-length of* $w \in W$. *We consider the left cosets of the stabilizer* W_ξ *in* W *and we denote by* ${}^\xi W \subset W$ *the set of* B^{σ_ξ}*-shortest representatives of the cosets. The set* ${}^\xi W$ *parametrizes the* B_ξ*-fixed points in* X *and thus the closed* L_ξ*-orbits,* $X^{B_\xi} = \{x_{w\sigma_\xi} : w \in {}^\xi W\}$.

Given $\lambda \in \Lambda^+$, *we denote, analogously to (5),*

$$\begin{aligned} &W = W^+(\sigma_\xi\lambda, \xi) \sqcup W^0(\sigma_\xi\lambda, \xi) \sqcup W^-(\sigma_\xi\lambda, \xi) \quad , \\ &W^+(\lambda, \xi) = \{w \in W : w\sigma_\xi\lambda(\xi) > 0\} \ , \\ &W^0(\lambda, \xi) = \{w \in W : w\sigma_\xi\lambda(\xi) = 0\} \ , \\ &W^-(\lambda, \xi) = \{w \in W : w\sigma_\xi\lambda(\xi) < 0\} \ . \\ &l^+_{\xi,\lambda} = \max\{l_\xi(w) : w \in {}^\xi W(\sigma_\xi\lambda, \xi)\} \ . \end{aligned}$$

Remark 4.2. *The situation is somewhat simpler with regard of calculations and notation whenever it suffices to consider one cubicle, which means that any Weyl chamber of* $\hat{G}$ *is contained in some Weyl chamber of* G, *listed as property (a) below. It is useful to notice that this property is preserved for the Levi subgroups. More generally, consider following the properties (note that (d)* $\Longrightarrow$ *(c)* $\Longrightarrow$ *(a)+(b)):*

(a) there exists a Weyl chamber of G *containing any given Weyl chamber of* $\hat{G}$, *i.e.,* $\hat{\Gamma}^+ \subset \Gamma^+$, *or equivalently* $W_{\mathrm{rel}} = \{1\}$;

(b) containing regular elements;

(c) $\hat{\Gamma}^{++} \subset \Gamma^{++}$, *i.e.,* $W_{\mathrm{com}} = \{1\}$ *and there is a single cubicle;*

(d) being a diagonal embedding of a semisimple group in a Cartesian power.

The definitions immediately imply the following:

1) If $\hat{G} \subset G$ has some of the properties (a),(b),(c),(d), then the same properties also hold for the natural embedding $\hat{L}'_\xi \subset L'_\xi$ of the semisimple parts of the centralizers of any semisimple element $\xi \in \hat{\mathfrak{g}}$.

2) Suppose that $G_1 \subset G_2 \subset G$ is a chain of embeddings. The embedding $G_1 \subset G$ has some of the properties (a),(b),(c),(d), if and only if the same properties hold for both embeddings $G_1 \subset G_2$ and $G_2 \subset G$.

Property (c) means, that the intrinsic notion of regularity for one-parameter subgroups of $\hat{G}$ coincides with that induced from the embedding in G, as mentioned above. Examples where this property is fulfilled are:
- *diagonal embeddings $\hat{G} \subset G = \hat{G}^{\times k}$;*
- *$SL_2 \subset SL_3$ given by any root;*
- *principal SL_2-subgroups $SL_2 \to G$ (characterized by having a single closed orbit in G/B);*
- *subgroup containing principal SL_2-subgroups $SL_2 \to \hat{G} \subset G$, for instance $Sp_{2\ell} \subset SL_{2\ell}$ and $SO_{2\ell+1} \subset SL_{2\ell+1}$.*

4.2.3 A formula for the unstable locus

Lemma 4.3. *Let $\mathfrak{P} = \{P_\xi \subset G : \xi \in \hat{\Gamma}^+ \setminus \{0\}\}$ denote the set of parabolic subgroups of G defined by nonzero dominant OPS of $\hat{G}$. Let $\mathfrak{P}_{\max}$ denote the set of maximal elements of $\mathfrak{P}$. Let $\{\xi_1, ..., \xi_q\} \subset \hat{\Gamma}^+$ denote the set of elements obtained as integral generators of rays of cubicles, and let $P_j = P_{\xi_j}$. Then $\mathfrak{P}_{\max} = \{P_1, ..., P_q\}$.*

Proof. Recall that the parabolic subgroups of G containing T are determined by their (simple) roots. Let us fix a cubicle and denote $\hat{\Gamma}^+_\sigma = \hat{\Gamma}^+ \cap \hat{\mathfrak{t}}_\sigma$. For any two elements $\xi, \eta \in \hat{\Gamma}^+_\sigma$ belonging to a fixed cubicle, and hence to the same Weyl chamber of G, we have $P_{\xi+\eta} \subset P_\xi \cap P_\eta$. Hence the parabolic subgroups defined by the generating rays of the cone $\hat{\mathfrak{t}}_\sigma$ are the maximal elements in the set of parabolic subgroups defined by $\xi \in \hat{\Gamma}^+_\sigma$. The elements of $\mathfrak{P}$ defined by elements of $\hat{\Gamma}^+_\sigma$ are characterized by the property that of containing the Borel subgroup B^σ. Hence, if $\xi \notin \hat{\Gamma}^+_\sigma$, then P_ξ does not contain P_η with $\eta \in \hat{\Gamma}^+_\sigma$. Since every element of $\hat{\Gamma}^+$ is contained in some cubicle, we obtain that the maximal elements of $\mathfrak{P}$ are exactly these defined by rays of cubicles. $\square$

We denote $\Xi_{\max} = \{\xi_1, ..., \xi_q\}$ and $\sigma_j = \sigma_{\xi_j}$.

Remark 4.3. *When a Weyl chamber of $\hat{G}$ is contained in a Weyl chamber of G, then the set $\Xi_{\max}$ consists simply of the fundamental coweights of $\hat{G}$, i.e., the generators of $\hat{\Gamma}^+$.*

Theorem 4.4. *For any $\lambda \in \Lambda^{++}$ the $\hat{G}$-unstable locus in $X = G/B$ can be written as*

$$X^{us}(\lambda) = \bigcup_{j=1}^{q} \hat{G} X^{us}_{\xi_j}(\lambda) = \hat{G} \bigcup_{j=1}^{q} \bigsqcup_{w \in {}^{\xi_j}W : w\sigma_j\lambda(\xi_j) > 0} P_j x_{w\sigma_j} .$$

The codimensions of the ξ_j-unstable locus and the $\hat{G}$-unstable locus are bounded from below by

$$\begin{aligned}\operatorname{codim}_X X^{us}_{\xi_j}(\lambda) &\geq r_j - \hat{r}_j - l^+_{j,\lambda} \ , \\ \operatorname{codim}_X X^{us}(\lambda) &\geq \min_j \{r_j - \hat{r}_j - l^+_{j,\lambda}\} \ ,\end{aligned}$$

where $l^+_{j,\lambda} = l^+_{\xi_j,\lambda}$ is the number given in Definition 4.1

Proof. From the Hilbert-Mumford criterion, we know that the $\hat{G}$-unstable locus is the $\hat{G}$-saturation of the union of the unstable loci for dominant OPS of $\hat{G}$. To reduce the instability with respect to an arbitrary $\xi \in \hat{\Gamma}^+$ to instability with respect to one of the ξ_j's we shall use Lemma 4.3. Let us take some $\xi \in \hat{\Gamma}^+ \setminus \{0\}$ and $x \in X^{us}_{\xi}(\lambda)$. The ξ-unstable locus is described by Lemma 4.1, and we conclude that $x \in P_\xi x_w$ for some $w \in W$ such that $w\lambda(\xi) > 0$. The element ξ belongs to some cubicle $\hat{\Gamma}^+_\sigma$ and can be expressed as a linear combination of the generators of this cubicle, say $\xi_1, ..., \xi_p$, with nonnegative coefficients. We can deduce that $w\lambda(\xi_j) > 0$ for some $j \in \{1, ..., p\}$ for which the coefficient of ξ is nonzero. Hence $P_\xi \subset P_j$ and we have $x \in P_\xi x_w \subset P_j x_w \subset X^{us}_{\xi_j}(\lambda)$. This proves the first formula for $X^{us}(\lambda)$. The second formula is deduced directly from Lemma 4.1 and Definition 4.1.

The bound on the codimension follows from the standard fact that, for any $\xi \in \hat{\Gamma}^+$, the parabolic subgroup $\hat{P}_\xi \subset \hat{G}$ satisfies $\hat{P}_\xi = P_\xi \cap \hat{G}$, hence it acts every P_ξ-orbit $P_\xi x$, and we have a surjective map

$$\hat{G} \times_{\hat{P}_\xi} P_\xi x \to \hat{G} P_\xi x \ .$$

The dimension of the fibre bundle is $\hat{r}_j + \dim P_\xi x$ and hence this number bounds the dimension of $\hat{G} P_\xi x$ from above. We take a maximal dimensional $P_\xi x_{w\sigma_\xi}$ inside $X^{us}_\xi(\lambda)$, with $w \in {}^\xi W$. We may apply the codimension formula of Lemma 4.1, part (ii), to the length function l_ξ referring to the cubicle of ξ (see Definition 4.1). We obtain $\operatorname{codim}_X P_\xi x_{w\sigma_\xi} = r_\xi - l_\xi(w) = r_\xi - l^+_{\xi,\lambda}$. This completes the proof. $\square$

We can deduce from the above proof the following expression for the $\hat{T}$-unstable locus. Note that unstable loci for tori are just a particular case of the above theorem, or, as well of the corollary. What we state below concerns Cartan subgroups of reductive groups and uses the Weyl group action on $X^{us}_{\hat{T}}(\lambda)$.

Corollary 4.5. *The $\hat{T}$-unstable locus for a given $\lambda \in \Lambda^{++}$ is given by*

$$X^{us}_{\hat{T}}(\lambda) = \bigcup_{\hat{w} \in \hat{W}} \bigcup_{j=1}^{q} X^{us}_{\hat{w}\xi_j}(\lambda) \ .$$

Its codimension is given by

$$\operatorname{codim}_X X^{us}_{\hat{T}}(\lambda) = \min\{r_j - l^+_{j,\lambda} : j = 1, ..., q\} \ .$$

4.3 The Kirwan-Ness stratification of the unstable locus

To compute the dimension of the unstable locus, we shall use the Kirwan-Ness stratification (Theorem 2.2) applied to the case $X = G/B$ embedded in $\mathbb{P}(V(\lambda))$ by the ample line bundle corresponding $\lambda \in \Lambda^{++}$. The one-parameter subgroups of any subgroup $\hat{G} \subset G$ are also one-parameter subgroups of G, and Lemma 4.1 provides a description of the resulting blades as orbits of parabolic subgroups of G. The lemma concerns $\xi \in \Gamma$ dominant with respect to the same Weyl chamber as λ. The set $\hat{\Gamma}^+$ of dominant OPS of $\hat{G}$ is partitioned by the cubicles, and for $\xi \in \hat{\Gamma}^+ \cap \hat{\mathfrak{t}}_\sigma$ we obtain

$$X^{\xi,m}(\lambda) = \bigsqcup_{w \in {}^\xi W : w\sigma\lambda(\xi) = m} L_\xi x_{w\sigma} \ , \quad X_{\xi,m}(\lambda) = \bigsqcup_{w \in {}^\xi W : w\sigma\lambda(\xi) = m} P_\xi x_{w\sigma} \ . \tag{6}$$

It remains to determine the stratifying pairs ξ, m. According to the stratification theorem, we consider the OPS determined by averaging weights of $\hat{T}$-fixed points in $X^{\xi,m}(\lambda)$. In our case the set of $\hat{T}$-weights is the projection of the set of T-weights, which in turn is a union of W_ξ-orbits:

$$St_T(X^{\xi,m}(\lambda)) = \bigsqcup_{w \in {}^\xi W : w\sigma\lambda(\xi) = m} W_\xi w\sigma\lambda \ , \quad St_{\hat{T}}(X^{\xi,m}(\lambda)) = \iota^*(St_T(X^{\xi,m}(\lambda))) \ . \tag{7}$$

The stratification theorem and the above observations bring us to the following definitions.

Definition 4.2. *(1) Let $\mathfrak{L} = \{L_\xi : \xi \in \hat{\Gamma}^+ \setminus \{0\}\}$ be the set of centralizers in G of nonzero dominant one-parameter subgroups of $\hat{T}$.*

(2) For any triple $(L, w, \lambda) \in \mathfrak{L} \times W \times \Lambda$, let $\nu_{L,w,\lambda} \in \hat{\Lambda}_{\mathbb{R}}$ denote the closest to 0 point in $\mathrm{Conv}(\iota^*(W_L w\lambda))$. If $\nu_{L,w,\lambda} \neq 0$, let $\xi_{L,w,\lambda} \in \hat{\Gamma}$ be the indivisible integral generator of the ray in $\hat{\mathfrak{t}}$ corresponding, under the Killing form, to the ray of $\nu_{L,w,\lambda}$ in $\hat{\Lambda}_{\mathbb{R}}$. In case $\nu_{L,w,\lambda} \neq 0$, we put $\xi_{L,w,\lambda} = 0$.

(3) For $\lambda \in \Lambda^+$, denote

$$\begin{aligned}
\Xi_\lambda &= \{\xi_{L,w,\lambda} \in \hat{\Gamma} : (L, w) \in \mathfrak{L} \times W\} \ , \\
\Xi_\lambda^+ &= \Xi_\lambda \cap \hat{\Gamma}^+ \ , \\
\Xi'_\lambda &= \{\xi \in \Xi_\lambda^+ \setminus \{0\} : L_\xi \in \mathfrak{L}'\} \ , \\
\Xi\mathfrak{W}_\lambda^+ &= \{(\xi, w) \in \Xi_\lambda^+ \times W : w \in {}^\xi W^+(\sigma_\xi\lambda, \xi)\} \ .
\end{aligned}$$

Remark 4.4. *Note that $\Xi_{\max} \subset \Xi_\lambda^+$ for every $\lambda \in \Lambda^{++}$. The Levi subgroups $L_j = L_{\xi_j}$, $j = 1, ..., q$, are exactly the maximal elements of $\mathfrak{L}$, i.e.,*

$$\mathfrak{L}_{\max} = \{L_\xi : \xi \in \Xi_{\max}\} \ . \tag{8}$$

We denote $L_j = L_{\xi_j}$ the Levi component of P_j for $j = 1, , , .q$. For every L_j, the intersection $Z(\mathfrak{l}_j) \cap \hat{\mathfrak{g}}$ of the center of $\mathfrak{l}_j$ with $\hat{\mathfrak{g}}$ is one-dimensional, generated by ξ_j. Thus, for every $(w, \lambda) \in W \times \Lambda$ the element $\xi_{L_j,w,\lambda}$ is proportional to ξ_j and in fact

$$\xi_{L_j,w,\lambda} \in \{\xi_j, 0, -\xi_j\} \ .$$

Lemma 4.6. *Let $\lambda \in \Lambda^{++}$. The set of dominant stratifying OPS for the $\hat{G}$-unstable locus $X^{us}(\lambda)$ is given by*

$$\mathfrak{S}_\lambda = \{\xi \in \Xi'_\lambda : \exists w \in W : w\sigma_\xi\lambda(\xi) > 0, w\sigma_\xi\lambda \in C^{\hat{L}'_\xi}(L'_\xi x_{w\sigma_\xi})\} \ .$$

Denote

$$\mathfrak{SW}_\lambda = \{(\xi, w) \in \mathfrak{S}_\lambda \times W : w \in {}^\xi W^+(\sigma_\xi\lambda, \xi), (L_\xi[v_{w\sigma_\xi\lambda}])^{ss}_{\hat{L}'_\xi} \neq \emptyset\}.$$

Then there is a natural bijective map $\mathfrak{SW}_\lambda \to \widetilde{\mathfrak{S}}_\lambda$ onto the set of stratifying pairs, given by $(\xi, w) \to (\xi, w\sigma_\xi\lambda(\xi))$.

Proof. The lemma follows from the Kirwan-Ness stratification theorem and Lemma 4.1. Indeed, we were led to the definition of the set Ξ'_λ by applying to our $X = G/B$ the constructions of the blades need in Definition 2.1 of stratifying elements. The property which remains to be checked is the presence of Levi-semistable points in the blades. Our blades are parametrized by $\Xi^+_\lambda \setminus \{0\}$, and we have to check the condition

$$(L_\xi[v_{w\sigma_\xi\lambda}])^{ss}_{\hat{L}_\xi/\xi} \neq \emptyset$$

with $\xi \in \Xi_\lambda$ and $w \in {}^\xi W^+(\lambda, \xi)$. The above condition may be expressed as: the restriction of the line bundle $\mathcal{L}_\lambda$ from X to $L_\xi x_{w\sigma_\xi}$ is $\hat{L}_\xi/\xi$-ample. The fact that ξ is of the form $\xi_{L,w,\sigma_\xi\lambda}$ ensures that the $\hat{T}/\xi$-semistable locus is nonempty, so the line bundle is $\hat{L}_\xi/\xi$-ample if and only if it is $\hat{L}'_\xi$-ample. Since w is the shortest representative in its left W_ξ-coset, the weight $w\sigma_\xi\lambda$ (or rather its appropriate restriction) is dominant with respect to the Borel subgroup $B^{\sigma_\xi} \cap L'_\xi$ of L'_ξ. So the requested semistable locus is nonempty if and only if $w\sigma_\xi\lambda \in C^{\hat{L}'_\xi}(L'_\xi/(B^{\sigma_\xi} \cap L'_\xi))$, which is just the condition imposed in the definition $\mathfrak{S}_\lambda$. For the second statement of the lemma it remains to notice that the requirement for w to be the shortest representative in its W_ξ-coset ensures a bijective correspondence between $\mathfrak{SW}_\lambda$ and the set of connected components of Kirwan-Ness strata. $\square$

Theorem 4.7. *Let $\lambda \in \Lambda^{++}$. The Kirwan-Ness stratification of $\hat{G}$-unstable locus in $X = G/B$ with respect to the line bundle $\mathcal{L}_\lambda$ is given by*

$$X^{us}(\lambda) = \bigsqcup_{(\xi,w)\in\mathfrak{SW}_\lambda} \hat{G}(P_\xi x_{w\sigma_\xi})^{ss}_{\hat{L}_\xi/\xi}(w\sigma_\xi\lambda) \ .$$

The dimension and codimension of the stratum for $(\xi, w) \in \mathfrak{SW}_\lambda$ are given by

$$\dim \hat{G}(P_\xi x_{w\sigma_\xi})^{ss}_{\hat{L}_\xi/\xi}(w\sigma_\xi\lambda) = \dim \hat{G}/\hat{P}_\xi + \dim P_\xi x_{w\sigma_\xi} = \hat{r}_\xi + n_\xi + l_\xi(w) \ ,$$
$$\operatorname{codim}_X \hat{G}P_\xi x_{w\sigma_\xi} = r_\xi - \hat{r}_\xi - l_\xi(w) \ .$$

The dimension and codimension of the unstable locus are given by

$$\dim X^{us}(\lambda) = \max\{\hat{r}_\xi + n_\xi + l^{\mathrm{str}}_{\xi,\lambda} : \xi \in \mathfrak{S}_\lambda\} \ ,$$
$$\operatorname{codim}_X X^{us}(\lambda) = \min\{r_\xi - \hat{r}_\xi - l^{\mathrm{str}}_{\xi,\lambda} : \xi \in \mathfrak{S}_\lambda\} \ ,$$

where $l^{\mathrm{str}}_{\xi,\lambda} = \max\{l_\xi(w) : (\xi, w) \in \mathfrak{SW}_\lambda\}$.

Proof. The formula for the unstable locus follows from Lemma 4.6. The dimension formulae follow from the Kirwan-Ness dimension formula for the strata and Lemma 4.1. $\square$

Remark 4.5. *In Section 4.5, based on ideas of Popov, [P03], we present an algorithm, using rooted trees, giving a "yes" or "no" answer to the question whether a given λ belongs to $C^{\hat{G}}(X)$. This algorithm can be applied to the Levi subgroups $\hat{L}'_\xi \subset L'_\xi$ and any given $w\sigma_\xi\lambda$, in order to determine completely the Kirwan-Ness stratification in any given case.*

4.4 The $\hat{G}$-ample cone of G/B and GIT-classes

The codimension formula for the $\hat{G}$-unstable locus of $\mathcal{L}_\lambda$ in X provides a description of the $\hat{G}$-ample cone $C^{\hat{G}}(X)$ by linear inequalities. In fact we have similar descriptions obtained simultaneously for all cones $C^{\hat{G}}_k(X)$ from Theorem III, showing that these form a sequence of rational polyhedral cones in $\Lambda^+_{\mathbb{R}}$, each contained in the relative interior of the previous one, in the relative topology of the Weyl chamber. We need some preliminary results on GIT-classes with respect to the torus $\hat{T}$, which form a subdivision of the GIT-classes for $\hat{G}$.

4.4.1 GIT-chambers for $\hat{T}$ and $\hat{G}$

Recall that a GIT-class on X with respect to a given subgroup of G is a chamber if the unstable locus is a proper subvariety and contains all points with positive dimensional stabilizer. In the flag variety, the connected components of the fixed point set of a one-parameter subgroup are the closed orbits of its centralizer. We have the following.

Lemma 4.8. *Let $\mathfrak{L}_{\max} = \{L_1, ..., L_q\}$ be the set of maximal elements in $\mathfrak{L}$ with respect to inclusion (see Remark 4.4). For any $\lambda \in \Lambda^{++}$ the following are equivalent:*

(i) *the $\hat{T}$-GIT-class of λ is a $\hat{T}$-chamber;*

(ii) $0 \notin \Xi_\lambda$*, i.e. the closed orbits $L_\xi x_w$ in G/B of centralizers in G of nontrivial OPS of $\hat{T}$ are $\hat{T}$-unstable;*

(iii) $\xi_{L_j,w,\lambda} \neq 0$ *for all $w \in W$ and $j = 1, ..., q$.*

(iv) $\lambda(w\xi_j) \neq 0$ *for all $w \in W$ and $j = 1, ..., q$.*

Proof. By Lemma 4.1 the fixed point set of any OPS is the union of the closed orbits of its centralizer. So, having a $\hat{T}$-chamber is equivalent to having all closed Levi orbits unstable, which is in turn equivalent to (ii).

To see that (iii) implies (ii) recall that $\xi_{L,w,\lambda} \in \hat{\Gamma}$ is the indivisible OPS corresponding to the weight $\nu_{L,w,\lambda} \in \hat{\Lambda}_{\mathbb{R}}$, which is the closest to 0 point in the convex hull of $\iota^*(W_L w\lambda)$, and hence if $\xi_{L,w,\lambda} = 0$ for some $L \in \mathfrak{L}$ and $w \in W$, then $\xi_{L_j,w,\lambda} = 0$ for any $L_j \supset L$.

The equivalence of (iii) and (iv) follows form Remark 4.4. $\square$

Theorem 4.9. *The decomposition of the ample cone $\Lambda^+_{\mathbb{R}}$ on X into GIT-classes with respect to $\hat{T}$ is defined by the following system of hyperplanes, parametrized by pairs $\xi \in \Xi_{\max}$, $w \in {}^{\xi}W$:*

$$\mathcal{H}_{\sigma_\xi^{-1}w^{-1}\xi} = \{\lambda \in \Lambda_{\mathbb{R}} : \lambda(\sigma_\xi^{-1}w^{-1}\xi) = 0\} ,$$

Proof. We described the $\hat{T}$-unstable locus for $\lambda \in \Lambda^{++}$ in Corollary 4.5. By Lemma 4.8 (specifically (i)$\Longleftrightarrow$(iii)), the walls bounding the $\hat{T}$-chambers are indeed defined by hyperplanes of the above form. $\square$

Theorem 4.10. *Let C be a $\hat{T}$-chamber consider a regular facet $F \subset \overline{C}$, which is necessarily of the form $\overline{F} = \overline{C} \cap \mathcal{H}_{\sigma_\xi^{-1}w^{-1}\xi}$ according to Theorem 4.9. Then C and F define distinct GIT-classes with respect to $\hat{G}$ if and only if $w\sigma_\xi F \subset C^{\hat{L}'_\xi}(L_\xi x_w)$.*

Proof. If the condition $w\sigma_\xi F \subset C^{\hat{L}'_\xi}(L_\xi x_w)$ implies, by Theorem 4.7, that $\hat{G}P_\xi x_w$ (or rather an open subset of it) is a Kirwan-Ness stratum for $\lambda \in F$. $\square$

Example 4.2. *The following example shows that the $\hat{G}$-classes do not coincide with the $\hat{T}$-classes. Consider the diagonal embedding $\hat{G} = SL_3 \hookrightarrow SL_3^{\times 3} = G$. Let $\lambda = (\hat{\lambda}, \hat{\lambda} + \hat{\lambda}^*, \lambda^*)$, with any $\hat{\lambda} = \binom{a_1}{a_2}$ satisfying $a_1 > a_2 > 0$, where a_j are the coordinates with respect to the fundamental weights of SL_3. Then we have $\lambda \in C^{\hat{G}}(X) \cap \mathcal{H}_{w^{-1}\hat{\epsilon}_1}$, where $w = (s_1 s_2, s_1, 1)$. The semisimple centralizer subgroups are then given by $\hat{L}'_1 = SL_2 \hookrightarrow SL_2^{\times 3} = L'_1$, and the variety $Z = L_1 x_w$ is a triple product $(\mathbb{P}^1)^{\times 3}$. The element w is of minimal length in its coset $W_1 w$, where $W_1 = \{1, s_2\}^{\times 3}$. Further, we calculate that*

$$\lambda_1 = w\lambda_{|\mathfrak{t}\cap\mathfrak{L}'_1} = \frac{1}{3}(a_1 - a_2, 3a_1 + 3a_2, 2a_1 + a_2) .$$

The middle coordinate of this weight, $a_1 + a_2$, exceeds the sum of the other two coordinates, which is a_1. Hence, from our knowledge of the SL_2-ample cone for diagonal embeddings, we deduce that $\lambda_1 \notin C^{\hat{L}'_1}(Z)$. (Formally, we describe the $\hat{G}$-maple cone on the next section, so this example could wait, but the case of $SL_2 \subset SL_2^{\times 3}$ this follows simply from the Clebsch-Gordon rule.) Hence $\hat{G}P_{\hat{\epsilon}_1}x_w$ is not a Kirwan-Ness stratum for λ and, by continuity, for weights in a neighbourhood of λ in $\Lambda_{\mathbb{R}}$.

4.4.2 The $\hat{G}$-ample and -movable cones

We have seen that the Kirwan-Ness strata of the unstable locus are of the form $\hat{G}P_\xi x_{w\sigma_\xi}$ for stratifying pairs $(\xi, w) \in \hat{\Gamma}^+ \times W$. The dimension of such a stratum is given by $\dim \hat{G}P_\xi x_{w\sigma_\xi} = \hat{r}_\xi + \dim P_\xi x_w$. It is convenient to consider the pairs (ξ, w) for which the dimension condition is satisfied; note that this property concerns just the action of $\hat{G}$ on G/B and does not refer to any λ.

Definition 4.3. *We call a pair $(\xi, w) \in \hat{\Gamma}^+ \times W$ of a dominant OPS of $\hat{G}$ and a Weyl group element of G a fit pair, if $\dim \hat{R}^-_\xi P_\xi x_{w\sigma_\xi} = \dim P_\xi x_{w\sigma_\xi} + \dim \hat{R}^-_\xi$.*

We denote the set of fit pairs, with the additional requirement that w is the B^{σ_ξ}-shortest element in its left W_ξ-coset, by

$$\Xi\mathfrak{W}_{\mathrm{fit}} = \{(\xi, w) \in \hat{\Gamma}^+ \times W : w \in {}^\xi W, \operatorname{codim}_X \hat{G} P_\xi x_{w\sigma_\xi} = r_\xi - \hat{r}_\xi - l_\xi(w)\} \ .$$

For fixed $\xi \in \hat{\Gamma}^+$, we denote by ${}^\xi W_{\mathrm{fit}}$ the set of elements in ${}^\xi W$ forming a fit pair (ξ, w); for $l \in \mathbb{N}$, we denote ${}^\xi W_{\mathrm{fit}}(l)$ the subset with $l_\xi(w) = l$.

Remark 4.6. *Let $\Delta = \Delta^+ \sqcup \Delta^-$ be a root system split into positive and negative parts. It is well known that a Weyl group element is uniquely determined by the set of positive roots it sends to negatives. For $w \in W$, the set $\Phi_w = \Delta^+ \cap w^{-1}\Delta^-$ is called the inversion set and the set $\Psi_w = \Delta^- \cap w\Delta^+$ the inverted set. We have $l(w) = \#\Psi_w$ and $\Psi_w = -\Phi_{w^{-1}}$. For a given $\xi \in \Gamma^+ \setminus \{0\}$, the decomposition $\Delta = \Delta(\mathfrak{l}_\xi) \sqcup \Delta(\mathfrak{r}_\xi^+) \sqcup \Delta(\mathfrak{r}_\xi^-)$ is invariant under the action of W_ξ. An element $\tau \in W_\xi$ is determined by its relative inverted set $\Delta^-(\mathfrak{l}_\xi) \cap w\Delta^+(\mathfrak{l}_\xi)$. Also, we have $\Psi_{\tau w} \cap \Delta(\mathfrak{r}_\xi^-) = \tau(\Psi_w \cap \Delta(\mathfrak{r}_\xi))$; in particular, these to sets have the same cardinality. Hence the shortest element in a coset $W_\xi w$ is characterized by the property $\Psi_w \subset \Delta(\mathfrak{r}_\xi^-)$, while the longest is characterized by $\Psi_w \supset \Delta^-(\mathfrak{l})$.*

Lemma 4.11. *Let $\Delta = \Delta^+ \sqcup \Delta^-$ be a root system split into positive and negative parts and let $w \in W$. Then there exists an order on the inverted set $\Psi_w = \{\beta_1, ..., \beta_l\}$ such that, upon setting $w_j = s_{\beta_j} \dots s_{\beta_l}$, for $j = 1, ..., l$ and $w_{l+1} = 1$, one gets*

$$w = w_1 = s_{\beta_1} \dots s_{\beta_l} \ , \ \Psi_{w_{j+1}} = \Psi_w \setminus \{\beta_1, ..., \beta_j\} \ , l(w_{j+1}) = l - j.$$

Moreover, the root β_j is simple for $w_j\Delta^+$.

Proof. We shall proceed by induction on the length $l = l(w)$. Let Π be the set of simple roots in Δ^+, so that $w\Pi$ is the set of simple roots in $w\Delta^+$. Let $\beta = \beta_1 \in w\Pi \cap \Delta^-$; such an element exists, as long as $l > 0$. Consider $w_2 = s_\beta w$. Note that $-\beta \in w\Delta^- \cap \Delta^+$, so that, if $U_{-\beta} \subset B$ denotes the one-parameter unipotent subgroup of the root $-\beta$, then $\overline{U_{-\beta} x_w} = U_{-\beta} x_w \sqcup \{x_{w_2}\}$ and so $\overline{Bx_w} \supset Bx_{w_2}$. Since β is simple for $w\Delta^+$, we have $s_\beta w \Delta^+ \cap w\Delta^- = \{-\beta\}$. Hence $\{\beta\} = s_\beta w\Delta^- \cap w\Delta^+ = (\Delta^- \setminus \Psi_{s_\beta w}) \cap \Psi_w$. On the other hand,

$$\begin{aligned} \Psi_{s_\beta w} &= s_\beta w \Delta^+ \cap \Delta^- \\ &= (s_\beta w\Delta^+ \cap \Delta^- \cap w\Delta^-) \cup (s_\beta w \Delta^+ \cap \Delta^- \cap w\Delta^+) \\ &= \emptyset \cup \Psi_{s_\beta w} \cap \Psi_w \subset \Psi_w \ . \end{aligned}$$

Thus $\Psi_w = \{\beta\} \cup \Psi_{w_2}$. By induction on l based on the trivial case $l = 0$, i.e., $w = 1$, we obtain the statement of the lemma. □

Lemma 4.12. *Let $(\xi, w) \in \Xi\mathfrak{W}_{\mathrm{fit}}$ be a fit pair and $l = l_\xi(w)$. Then there exists a sequence $w = w_1, ..., w_{l+1} = 1$ in ${}^\xi W_{\mathrm{fit}}$ with $l_\xi(w_{j+1}) = l - j$ and*

$$\overline{P_\xi x_{w_j\sigma_\xi}} \supset P_\xi x_{w_{j+1}\sigma_\xi} \quad , \quad \operatorname{codim}_X \hat{G} P_\xi x_{w_j\sigma_\xi} = r_\xi - \hat{r}_\xi - l + j + 1 \ .$$

If $\lambda \in \Lambda^{++}$ and $w\sigma_\xi\lambda(\xi) \geq 0$, then $w_j\sigma_\xi\lambda(\xi) > 0$ for $j \geq 2$.

Proof. The proof is based on Lemma 4.11. We apply it to w with respect to the system of positive roots $\sigma_\xi \Delta^+$ associated to ξ. For the first part of the lemma, we may assume, without loss of generality, that $\sigma_\xi = 1$, so that $\xi \in \Gamma^+ \cap \hat{\Gamma}^+$. The case $l = 0$ being trivial, we assume $l \geq 1$. Note that the condition $w \in {}^\xi W$ is equivalent to $\Psi_w \subset \Delta(\mathfrak{r}_\xi^-)$. Hence, if $w = w_1, ..., w_{l+1} = 1$ is a sequence as obtained from Lemma 4.11, then $\emptyset \subset \Psi_{w_l} \subset ... \subset \Psi_{w_2} \subset \Psi_w \subset \Delta(\mathfrak{r}_\xi^-)$. In particular, all w_j belong necessarily to ${}^\xi W$. It follows that the R_ξ^--stabilizers of the points x_{w_j} are nested, i.e.,

$$1 = (R_\xi^-)_{x_1} \subset (R_\xi^-)_{x_{w_l}} \subset ... \subset (R_\xi^-)_{x_w} \ .$$

The pair (ξ, w_j) is fit if and only if the generic $\hat{R}_\xi^-$-stabilizer on $L_\xi x_{w_j}$ is trivial. Recall that $\hat{R}_\xi^- \subset R_\xi^-$, and also that the R_ξ^--stabilizers on $L_\xi x_w$ are L_ξ-conjugate, so L_ξ-conjugate to $(R_\xi^-)_{x_w}$. Thus the above chain of inclusions implies that, on each $L_\xi x_{w_j}$, the generic $\hat{R}_\xi^-$-stabilizer is trivial. Hence $w_j \in {}^\xi W_{\mathrm{fit}}(l - j + 1)$. This proves the first statement.

For the second statement of the lemma, let us use the notation from the proof of Lemma 4.11 and note that

$$s_\beta w \sigma_\xi \lambda = -(\textit{positive number})\beta(\xi) + w\sigma_\xi\lambda(\xi) > w\sigma_\xi\lambda(\xi) \ .$$

$\square$

Theorem 4.13. *For $k \geq 1$, $C_k^{\hat{G}}(X) = \overline{\{\lambda \in \Lambda_{\mathbb{Q}}^{++} : \mathrm{codim}_X X^{us}(\lambda) \geq k\}} \subset \Lambda_{\mathbb{R}}^+$ is a rational polyhedral cone. Whenever it is nonempty, it is given by:*

$$C_k^{\hat{G}}(X) = \{\lambda \in \Lambda_{\mathbb{R}}^+ : \ \lambda(\sigma_j^{-1} w^{-1} \xi_j) \leq 0 \ , \ w \in {}^{\xi_j} W_{\mathrm{fit}}(r_j - \hat{r}_j - k + 1), j = 1, ..., q\} \ .$$

Whenever the cone defined by the above inequalities belongs to $\partial \Lambda_{\mathbb{R}}^+$, the cone $C_k^{\hat{G}}(X)$ is empty. Furthermore, in the relative topology of $\Lambda_{\mathbb{R}}^+$, we have $C_{k+1}^{\hat{G}}(X) \subset \mathrm{Int}\, C_k^{\hat{G}}(X)$.

In particular, the $\hat{G}$-ample and $\hat{G}$-movable cones, if nonempty, are obtained for $k = 1, 2$ as

$$C^{\hat{G}}(X) = \{\lambda \in \Lambda_{\mathbb{R}}^+ : {}^{\xi_j} W_{\mathrm{fit}}(r_j - \hat{r}_j) \subset W^{0-}(\sigma_j \lambda, \xi_j) \ , \ \forall j\} \ ,$$
$$\mathrm{Mov}^{\hat{G}}(X) = \{\lambda \in \Lambda_{\mathbb{R}}^+ : {}^{\xi_j} W_{\mathrm{fit}}(r_j - \hat{r}_j + 1) \subset W^{0-}(\sigma_j \lambda, \xi_j) \ , \ \forall j\} \ .$$

Proof. In our description of the Kirwan-Ness stratification of the unstable locus in Theorem 4.7, for any given $\lambda \in \Lambda^{++}$, we have described the set of stratifying pairs in terms of the set $\mathfrak{S}\mathfrak{W}_\lambda$, which is a subset of the set $\Xi\mathfrak{W}_\lambda^+$ (see Def. 4.2) parameterizing the unstable parabolic orbits defined by dominant OPS of $\hat{G}$. We have

$$X^{us}(\lambda) = \bigcup_{(\xi, w) \in \Xi\mathfrak{W}_\lambda^+} \overline{\hat{R}_\xi^- P_\xi x_{w\sigma_\xi}} \ .$$

The dimension formula for the strata derived from the general stratification Theorem 2.2 implies that all stratifying pairs $(\xi, w) \in \mathfrak{S}\mathfrak{W}_\lambda$ are fit. Since the maximum dimension is attained at a stratifying element, we deduce that

$$\mathrm{codim}_X X^{us}(\lambda) = \min\{r_\xi - \hat{r}_\xi - l(w) : (\xi, w) \in \Xi\mathfrak{W}_\lambda^+ \cap \Xi\mathfrak{W}_{\mathrm{fit}}\} \ .$$

Then the cone $C_k^{\hat{G}}(X)$ can be characterized being generated by those λ whose unstable locus does not contain any $(k-1)$-codimensional parabolic orbit $P_\xi x_{w\sigma_\xi}$ corresponding to a fit pair (ξ, w). This means $w\sigma_\xi\lambda(\xi) \leq 0$ for all fit (ξ, w) with $w \in {}^\xi W(r_\xi - \hat{r}_\xi - k + 1)$. This proves description of $C_k^{\hat{G}}(X)$ given in the theorem.

To prove the second statement, we shall use the following lemma.

Lemma 4.14. *Suppose that $C_1, C_2 \subset \Lambda_{\mathbb{R}}^+$ are $\hat{T}$-GIT-classes intersecting the interior of the Weyl chamber.*

(a) If $\overline{C_1} \supset C_2$, then

$$0 \leq \mathrm{codim}_X X^{us}(C_2) - \mathrm{codim}_X X^{us}(C_1) \leq 1 \ .$$

(b) If C_1, C_2 are GIT-chambers sharing a facet C_{12}, then

$$|\mathrm{codim}_X X^{us}(C_1) - \mathrm{codim}_X X^{us}(C_2)| \leq 1 \ .$$

Proof. Suppose $\overline{C_1} \supset C_2$. Then we have $X^{us}(C_1) \supset X^{us}(C_2)$. In particular, $\mathrm{codim}_X X^{us}(C_1) \leq \mathrm{codim}_X X^{us}(C_2)$.

The change in the unstable locus $X^{us}(\lambda)$ as λ passes from C_1 to C_2 is necessarily reflected in a change of the set of stratifying elements $\mathfrak{SW}_\lambda$. By the continuity of the bilinear pairing between $\Lambda_{\mathbb{R}}$ and $\Gamma_{\mathbb{R}}$, there exists a stratifying pair (ξ, w), with $w \in {}^\xi W^+(C_1, \xi) \cap {}^\xi W^0(C_2, \xi)$. For any such (ξ, w), we have $(\xi, w) \in \Xi\mathfrak{W}_{\text{fit}}$ and we can apply Lemma 4.12. The element w_2 produced by that lemma is fit for ξ, satisfies $w_2 \in {}^\xi W^+(C_2, \xi)$ and has length $l(w) - 1$. Thus

$$\hat{G}P_\xi x_{w_2\sigma_\xi} \subset X^{us}(C_2) \quad , \quad \mathrm{codim}_X \hat{G}P_\xi x_{w_2\sigma_\xi} = \mathrm{codim}_X \hat{G}P_\xi x_{w\sigma_\xi} + 1 \ .$$

Therefore, the inequality

$$\mathrm{codim}_X \hat{G}X_\xi^{us}(C_2) - \mathrm{codim}_X \hat{G}X_\xi^{us}(C_1) \leq 1$$

holds for any ξ. Hence it also holds for the codimensions of the entire unstable loci. This proves part (a).

For part (b) a similar argument works. If the unstable locus changes for some $\xi \in \Xi_{\max}$, then there exists $w \in {}^\xi W^+(C_1, \xi) \cap {}^\xi W^0(C_{12}, \xi) \cap {}^\xi W^-(C_2, \xi)$. Then we have $w_2 \in {}^\xi W^+(C_{12}, \xi)$ and may we proceed as above. □

Form the lemma we deduce that the regular faces of $C_k^{\hat{G}}(X)$ give k-dimensional unstable loci, since they are contained in the closure of some GIT-chambers from $C_{k-1}^{\hat{G}}(X)$. This implies that the chambers contained in $C_k^{\hat{G}}(X)$, whose closure intersect the regular boundary of $C_k^{\hat{G}}(X)$, must also have k-dimensional unstable loci. These chambers form then a "layer" isolating $C_{k+1}^{\hat{G}}(X)$ from the regular boundary of $C_k^{\hat{G}}(X)$. This completes the proof of the theorem. □

Example 4.3. *It is not hard to show that for any SL_2-subgroup of SL_3 one has $C^{\hat{G}}(X) = \Lambda_{\mathbb{R}}^+$ and $\mathrm{Mov}^{\hat{G}}(X) = \emptyset$. Indeed, there are two conjugacy classes of SL_2-subgroups, but their Cartan subalgebras coincide, up to conjugacy, as vector spaces $\hat{\mathfrak{t}} \subset \mathfrak{t}$ (endowed however with different lattices $\hat{\Gamma}$). We have $\hat{\mathfrak{t}} = \mathbb{R}\rho^\vee$, and it* *suffices to evaluate weights on $\xi = \rho^\vee = \alpha_1^\vee + \alpha_2^\vee$, which is regular, so $\mathfrak{r}_\xi = \mathfrak{n} = 3$, $\hat{r}_\xi = 1$. Since $\hat{L}_\xi \cong \mathbb{C}^*$ and $\hat{L}'_\xi = 1$, all pairs $(\xi, w) \in \Xi\mathfrak{W}^+_\lambda$ are stratifying. A simple computation yields*

$$\Lambda^{++} = \{\lambda \in \Lambda : W^+(\lambda, \xi) = W(l \leq 1) = \{1, s_1, s_2\}\} \ , \ X^{us}(\lambda) = \hat{G}B_{x_{s_1}} \cup \hat{G}B_{x_{s_2}} \ .$$

Thus the facets of $C^{\hat{G}}(X)$ constructed by our theorem coincide with the walls of the Weyl chamber, and the interior constitutes a single GIT-chamber with unstable locus of codimension 1.

Since we shall be interested in $\hat{G}$-movable chambers, we record the following immediate corollary.

Corollary 4.15. *If $C_k^{\hat{G}}(X) \neq \emptyset$ for some $k \geq 2$, then X admits GIT-chambers where the unstable locus has codimension $k-1$. In particular, if there exists $\lambda \in \Lambda^{++}$ with $\mathrm{codim}_X X^{us}(\lambda) > 2$, then X admits $\hat{G}$-movable chambers.*

Remark 4.7. *In our previous work, [ST15], we have considered the case where $\hat{G}$ if a principal SL_2-subgroup G. We have shown that $\hat{G}$-movable chambers exist, except for a small number of degenerate cases for G. Under some more assumptions, e.g. G not having simple factors of rank 1 or 2, the entire ample cone is $\hat{G}$-movable.*

Remark 4.8. *There is an obvious upper bound for the codimension of the unstable locus, which can be deduced from the above theorem:*

$$\max\{\mathrm{codim}_X X^{us}(\lambda) : \lambda \in \Lambda^{++}\} \leq \min\{r_\xi - \hat{r}_\xi : \xi \in \Xi_{\max}\} \ .$$

In particular, we have $C^{\hat{G}}(X) = \emptyset$, when this upper bound is zero. This holds for instance for the natural embedding $\hat{G} = Sp_{2n} \subset SL_{2n} = G$, as well as in case $\hat{\mathfrak{g}}$ contains simple ideals of $\mathfrak{g}$.

Example 4.4. *Let us consider the case $\lambda = \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$, the smallest strictly dominant weight. We shall estimate the codimension of $X^{us}(\rho)$ in terms of invariants of the embedding $\hat{G} \subset G$ and thus give a criterion for existence of $\hat{G}$-movable chambers. We shall give some more precise calculations for diagonal embeddings $\hat{G} \subset \hat{G}^{\times k} = G$, where $\rho = (\hat{\rho}, ..., \hat{\rho})$, specifically for $\hat{G} = SL_m$.*

Let us begin with the general remark, that for $w \in W$, we have $w\rho = \frac{1}{2}(\langle \Phi^c_{w^{-1}} \rangle - \langle \Phi_{w^{-1}} \rangle)$, where $\langle \Phi \rangle$ denotes the sum of the elements of any subset $\Phi \in \Lambda$. Evaluated at any $\xi \in \Gamma^+$ this gives

$$w\rho(\xi) = \frac{1}{2}(\langle \Phi^c_{w^{-1}} \rangle - \langle \Phi_{w^{-1}} \rangle)(\xi) = \frac{1}{2}(\langle \Phi^c_{w^{-1}} \cap \Delta(\mathfrak{r}_\xi) \rangle - \langle \Phi_{w^{-1}} \cap \Delta(\mathfrak{r}_\xi) \rangle))(\xi) \ .$$

Since $\Phi^c_{w^{-1}} = \Phi_{w_0 w^{-1}}$, we conclude that either $w\rho(\xi) = ww_0\rho(\xi) = 0$, or exactly one of w and ww_0 belongs to $W^+(\rho, \xi)$ while the other one belongs to $W^-(\rho, \xi)$. Also $w \in {}^\xi W$ if and only if $\Phi_{w^{-1}} \subset \Delta(\mathfrak{r}_\xi)$. Put

$$a_\xi = \min\{\alpha(\xi) : \alpha \in \Delta(\mathfrak{r}_\xi)\} \quad b_\xi = \max\{\alpha(\xi) : \alpha \in \Delta(\mathfrak{r}_\xi)\}. \tag{9}$$

Then,

$$a_\xi(r_\xi - l(w)) - b_\xi l(w) \leq 2w\rho(\xi) \leq b_\xi(r_\xi - l(w)) - a_\xi l(w)\ .$$

It follows that, for $w \in {}^\xi W$,

$$l(w) < \frac{a_\xi r_\xi}{a_\xi + b_\xi} \implies w \in {}^\xi W^+(\rho,\xi)\ ;$$
$$l(w) \geq \frac{b_\xi r_\xi}{a_\xi + b_\xi} \implies w \in {}^\xi W^{0-}(\rho,\xi)\ .$$

Hence

$$\frac{a_\xi r_\xi}{a_\xi + b_\xi} - 1 \leq l^+_{\xi,\rho} < \frac{b_\xi r_\xi}{a_\xi + b_\xi}$$

and

$$\mathrm{codim}_X X^{us}(\rho) \geq \min_{\xi\in\Xi_{\max}} \{r_\xi - \hat{r}_\xi - l^+_{\xi,\rho}\} > \min_{\xi\in\Xi_{\max}} \{\frac{a_\xi}{a_\xi + b_\xi} r_\xi - \hat{r}_\xi\}\ .$$

We can use this concrete case, where the codimension is expressed in terms of structural invariants of the embedding $\hat{G} \subset G$, to obtain the following general criterion for existence of $\hat{G}$-movable chambers.

Proposition 4.16. *Given an embedding $\hat{G} \subset G$, if $\min_{\xi\in\Xi_{\max}} \{\frac{a_\xi}{a_\xi+b_\xi} r_\xi - \hat{r}_\xi\} \geq 2$ (cf. (9)), then the $\hat{G}$-ample cone on X admits $\hat{G}$-movable chambers.*

Let us consider now a diagonal embedding $\hat{G} \subset \hat{G}^{\times k} = G$, Any Weyl chamber of $\hat{G}$ is contained, as a diagonal, in a Weyl chamber of $\hat{G}$. The (maximal) Levi subgroups of G defined by nonzero elements of $\hat{\Gamma}^+$ are the k-fold products of (maximal) Levi subgroups of $\hat{G}$. We have $r_\xi = k\hat{r}_\xi$. The OPS defining the maximal Levi subgroups are the fundamental coweights $\Xi_{\max} = \{\hat{\xi}_1, ..., \hat{\xi}_{\hat{\ell}}\}$. Furthermore, for $\xi_j \in \Xi_{\max}$ we have $\hat{a}_{\xi_j} = \hat{a}_{\xi_j} = 2$, $b_{\xi_j} = \hat{b}_{\xi_j} = 2m_j$, where m_j is the j-th coefficient of the highest root of $\hat{G}$ expressed as a sum of simple roots, i.e., $\tilde{\alpha} = \sum m_j \hat{\alpha}_j$. Hence

$$\mathrm{codim}_X X^{us}_{\xi_j}(\rho) > \frac{1}{1+m_j} k\hat{r}_j - \hat{r}_j\ .$$

We can also see that the codimension of the unstable locus tends to ∞ when $k \to \infty$. Concerning $\hat{G}$-movable chambers, one can easily calculate that

$$k \geq \max\{\frac{\hat{r}_j + 2}{\hat{r}_j}(1 + m_j) : j = 1, ..., \hat{\ell}\} \implies \mathrm{codim}_X X^{us}(\rho) > 2$$
$$\implies \exists\, \hat{G} - \textit{movable chambers}\ .$$

In particular, one can deduce the following.

Proposition 4.17. *If $\hat{G}$ is a product of classical groups and $\hat{G} \subset G = \hat{G}^{\times k}$ is a diagonal embedding with $k \geq 5$, then the $\hat{G}$-ample cone admits $\hat{G}$-movable chambers.*

Example 4.5. *Let us consider the case $\hat{G} = SL_{\hat{\ell}+1}$, where $m_j = 1$ for all j. Then the above bound means that there are $\hat{G}$-movable chambers for $k > 2\frac{\hat{\ell}+2}{\hat{\ell}}$. Let us Going back a few steps, we compute, $w \in {}^{\xi_j}W$ we have*

$$w\rho(\xi_j) = \frac{1}{2}(\langle \Phi^c_{w^{-1}}\rangle - \langle \Phi_{w^{-1}}\rangle)(\xi_j) = \frac{1}{2}(r_j - l(w) - l(w)) = \frac{1}{2}r_j - l(w) = \frac{k}{2}\hat{r}_j - l(w) \ .$$

Thus $l^+_{j,\lambda} = \lfloor \frac{r_j - 1}{2} \rfloor$.

$$\mathrm{codim}_X \hat{G} X^{us}_{\xi_j}(\rho) \geq r_j - \hat{r}_j - l^+_{j,\lambda} = \lceil \frac{r_j + 1}{2} \rceil - \hat{r}_j = \lceil \frac{k\hat{r}_j + 1}{2} \rceil - \hat{r}_j = \lceil \frac{(k-2)\hat{r}_j + 1}{2} \rceil \ .$$

The minimum value over $j = 1, ..., \hat{\ell}$ is attained at $j = 1$, where $\hat{r}_j = \hat{\ell}$ and

$$\mathrm{codim}_X X^{us}(\rho) = \mathrm{codim}_X \hat{G} X^{us}_{\xi_1}(\rho) \geq \lceil \frac{(k-2)\hat{\ell} + 1}{2} \rceil \ .$$

We obtain $\mathrm{codim}_X X^{us}(\rho) > 2$ *except in the following cases:*

$$\mathrm{codim}_X \hat{G} X^{us}(\rho) = \begin{cases} 1 \ , \ if & k = 2 \\ 1 \ , \ if & k = 3, \hat{\ell} = 1 \\ 2 \ , \ if & k = 3, \hat{\ell} = 2, 3 \\ 2 \ , \ if & k = 4, \hat{\ell} = 1 \end{cases}$$

In particular, in all cases except the above, $\hat{G}$-movable chambers do exist.

4.5 Popov's tree-algorithm

Here we present an algorithm allowing to determine whether a given $\lambda \in \Lambda^{++}$ belongs to $C^{\hat{G}}(X)$ or not. Having in mind our description of Kirwan stratification of $X^{us}(\lambda)$, where the non-emptiness of the proposed strata depends on whether certain W-translate $w\sigma_\xi\lambda$ belongs to $C^{\hat{L}'_\xi}(L_\xi x_{w\sigma_\xi})$, this algorithm can be applied to determine the entire stratification. The idea is due to Popov, [P03], who developed the method in his study of unstable points in a linear representation space of a reductive group, the classical nullcone of a representation. There is a common generalization of his and our settings, where $X = G/P$ is a partial flag variety, $\mathbb{P}(V) = SL(V)/P_1$ in the classical case, with an action of a reductive subgroup $\hat{G} \subset G$. One particular feature of complete flag varieties, as well as of projective spaces, is that the closed orbits of Levi subgroups $L \subset G$ are of the same type, i.e., complete flag varieties, or, respectively, projective spaces. This is important, since the algorithm uses recursion, whose step refers to Levi subgroups acting on their closed orbits in X. This latter fact remains somewhat hidden in the classical case, where one considers linear subspaces of a vector space without necessarily mentioning Levi subgroups of its linear group.

Let $\mathfrak{Z}_X = \{L_\xi x_w : \xi \in \hat{\Gamma}, w \in W\}$ denote the set of closed orbits in X of Levi subgroups of G defined by one-parameter subgroups of $\hat{T}$. Next we define a rooted tree $\mathcal{T}_\lambda = T_{\hat{G},X,\lambda}$, whose vertices are associated to certain elements of $\mathfrak{Z}_X$.

The tree has a natural orientation and signature, which allow to determine, by a recursive algorithm, whether λ defines a $\hat{G}$-ample line bundle on X or not.

Every rooted tree is endowed with a natural orientation of the edges, pointing from the root to its adjacent vertices, and defined inductively for the rest of the tree.

Definition 4.4. *Let $\lambda \in \Lambda^{++}$. We denote*

$$\mathfrak{M}_X = \mathfrak{M}_{\hat{G},X,\lambda} = \{L_\xi x_{w\sigma_\xi} \in \mathfrak{Z}_X : \xi = \xi_{L_\xi,w,\sigma_\xi\lambda} \in \hat{\Gamma}^+\backslash\{0\}, w \in {}^\xi W, l_\xi(w) = r_\xi - \hat{r}_\xi\} .$$

Analogously, for any $Z = L_\xi x_w \in \mathfrak{Z}_X$, endowed with the action of $\hat{L}'_\xi$ and the line bundle given by $w\sigma_\xi\lambda$, we denote

$$\mathfrak{M}_Z = \mathfrak{M}_{\hat{L}'_\xi,Z,w\sigma_\xi\lambda} .$$

We define a rooted tree $\mathcal{T}_\lambda$ with vertices $a_{(Z_j)}$ associated to sequences of nested elements $(Z_j) = (Z_0 \supset Z_1 \supset \cdots \supset Z_p)$ of $\mathfrak{Z}_X$, starting at $Z_0 = X$, and satisfying $Z_{j+1} \in \mathfrak{M}_{Z_j}$. The root of $\mathcal{T}_\lambda$ is $a_{(X)}$. The vertices adjacent to $a_{(X)}$ are $a_{(X\subset Z)}$ for $Z \in \mathfrak{M}_X$. The vertices stemming from $a_{X\supset Z_1\supset\cdots\supset Z_p\supset Z}$ are, by definition, $a_{X\supset Z_1\supset\cdots\supset Z_p\supset Z}$ for $Z \in \mathfrak{M}_{Z_p}$.

The height of a vertex a is defined as the maximum length of an oriented path in $\mathcal{T}_\lambda$ starting at a.

A signature on the tree $\mathcal{T}_\lambda$ is defined as follows: a vertex a is given a sign "$-$" if there exists an arrow in $\mathcal{T}_\lambda$ emanating at a and ending at a vertex b with $sign(b) = +$; otherwise, a is given a sign "$+$".

Remark 4.9. *(i) The vertices of height 0, called the leaves, always have sign "$+$".*

(ii) If $\hat{G}$ is abelian, then the tree associated to any $\lambda \in \Lambda^{++}$ consists only of the root, $\mathcal{T}_\lambda = \{a_{(X)}\}$. Hence the sign is always "$+$", which corresponds to the fact that $C^{\hat{G}}(X) = \Lambda^+_{\mathbb{R}}$.

(iii) The maximal height of a vertex in the tree $\mathcal{T}_\lambda$ is the height of the root. It does not exceed $\operatorname{rank}(\hat{G})$*, since for chains $X \supset \ldots \supset Z$ of that length, or higher, the semisimple part of the Levi subgroup $\hat{L} \subset \hat{G}$ preserving Z is abelian.*

Theorem 4.18. *Let $\lambda \in \Lambda^{++}$. The line bundle $\mathcal{L}_\lambda$ on X is $\hat{G}$-ample if and only if the root of $\mathcal{T}_\lambda$ has sign plus, i.e.,*

$$C^{\hat{G}}(X) \cap \Lambda^{++} = \{\lambda \in \Lambda^{++} : sign(a_{(X)}) = + \text{ in } \mathcal{T}_\lambda\} .$$

Proof. We follow the idea of Popov, [P03]. Let us remark that the branches of $\mathcal{T}_\lambda$ are again trees of the same type. More precisely, let $a = a_{(Z_j)}$ be a vertex in $\mathcal{T}_\lambda$ and let $Z = Z_p$ be the last variety in the sequence defining a. Let $(\xi, w) \in \Xi^+_\lambda \times W$ be the elements associated to Z, λ according to the above definition. Then the branch of $\mathcal{T}_\lambda$ starting at a is identical with the tree $\mathcal{T}_{\hat{L}'_\xi,Z,w\sigma_\xi\lambda}$. This tree depends only on Z and λ, but not on the sequence (Z_j) connecting X to Z; we shall denote it by $\mathcal{T}_\lambda(Z)$.

We shall prove the theorem by induction on the height of the whole tree, i.e., the height of the root. In the above remark, we noticed that the vertices of height 0 always have sign plus. The height of the root $a_{(X)}$ is 0 if and only if $\mathfrak{M}_X = \emptyset$. The latter implies, via Theorem 4.7, that there are no Kirwan-Ness strata in $X^{us}(\lambda)$ of codimension 0, hence $\lambda \in C^{\hat{G}}(X)$. Thus the statement of the theorem holds in the base case. Assume it holds for trees of height one less than the height of $\mathcal{T}_\lambda$. The sign of the root $a_{(X)}$ is minus if and only if there is an adjacent vertex $a_{X \supset Z}$ with $Z \in \mathfrak{M}_X$ and sign plus. This means that the root of $\mathcal{T}_\lambda(Z)$ has sign plus. By hypothesis, this is equivalent to $w\sigma_\xi\lambda \in C^{\hat{L}'_\xi}(Z)$, In such a case (ξ, w) is a stratifying pair for $X^{us}_{\hat{G}}(\lambda)$ and, since $Z \in \mathfrak{M}_X$, we have $\mathrm{codim}_X \hat{G}P_\xi x_{w\sigma_\xi} = 0$. This is in turn equivalent to $\lambda \notin C^{\hat{G}}(X)$. $\square$

Example 4.6. *It is not hard to show that, for $\hat{G}$ of rank 1 or 2, a given $\lambda \in \Lambda^{++}$ belongs to $C^{\hat{G}}(X)$ if and only if $\mathcal{T}_\lambda$ does not have branches of length 1. For* $\mathrm{rank}(\hat{G}) = 1$*, this means $\mathcal{T}_\lambda = \{a_{(X)}\}$.*

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